

Stochastic Model Simplification

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Abstract—Mathematical models, defined by a structure and a set of parameter variation or uncertainty, may be simplified both by structure reduction and parameter set reduction. First, the approximation of high-order and time varying linear Gaussian models by low-order and time-invariant ones is considered. The proposed approach is based on maximizing the probabilistic ambiguity between the actual model and the approximate one, and is applicable to general stochastic linear models. Reducing a model set, defined on a set of parameter variation or uncertainty, to a single fixed parameter model or a finite model group, is then considered. The set reduction criteria give rise to a min-max optimization problem and a min-max-min problem, which is converted to a constrained min-max problem. The algorithmic solution of the optimization problems is considered in detail, along with several approximation and discretization schemes. The application and the validity of the proposed approach are examined in view of traditional design considerations by solving numerical examples for several structure and set reduction problems.

I. INTRODUCTION

THIS paper is concerned with the problem of simplifying high-order, time-varying and parameter-uncertain stochastic linear models. Order reduction methods for essentially deterministic, time-invariant linear models have been studied extensively in recent years [1]. Various techniques, ranging from dominant eigenvectors [2], canonical forms [3] and output deviation [4], approximations of state-space models to continued fraction [5], state feedback [6], Pade [7], [8], and Routh [9], approximations of transfer functions, have been suggested. Stochastic model reduction has been considered, in rather general terms, in [10] and [11]. On the other hand, with very few exceptions (e.g., [12], and [13], where only response to initial conditions is considered) the problem of representing time-varying models by time-invariant ones does not seem to have attracted much interest. Neither has the problem of representing models with large parameter variation or uncertainty by a single model or a small model group received a systematic treatment. This is somewhat surprising, since simplified modeling of systems, operating under varying conditions, is needed in many applications (e.g., in aerospace control and navigation systems) for analysis and design purposes.

In this paper we present a unified approach to the model simplification problem. The proposed approach is based on maximizing the expected log likelihood function or the

probabilistic ambiguity between the actual system and the simplified model. An approximation coefficient, which proves to be very useful in the analysis, is also defined. Employing state-space models in the time domain makes it possible to consider a general class of linear model approximation problems. In particular, a single design would take account of stochastic disturbances, deterministic inputs, and initial conditions, which have been treated previously in the order reduction literature by specialized techniques. For the case of time-invariant systems, transient response and steady-state approximations are naturally defined in the time domain and prove to be compatible with traditional dominant pole approximations.

The problems of (order and time) structure reduction and of (parameter) set reduction give rise to different programming problems. While structure reduction is achieved by solving a minimization problem, set reduction to a single model requires solving a min-max problem and set reduction to a model group requires solving a min-max-min problem. The algorithmic aspects of these problems are discussed in detail. The validity and the usefulness of the proposed techniques are demonstrated by low-order numerical examples. The resulting designs, based on our purely statistical criterion are found to be in complete agreement with what would be obtained from system-theoretic considerations and "classical" designs in the frequency domain.

II. STRUCTURE (ORDER AND TIME) REDUCTION

A. The Approximation Criterion

1) *General*: Consider a finite sequence of observations $Y^N = (y_1, \dots, y_N)$ with $y_i \in R^l$. Let $\bar{M} = \{M(s); s \in S\}$ denote a set of arbitrarily simple (hence, not necessarily accurate) models for Y^N , depending on a parameter s . A widely accepted criterion for selecting a model for Y^N from \bar{M} is the maximum likelihood criterion

$$\max_{s \in S} \log f_s(Y^N) \quad (2.1)$$

where $f_s(Y^N)$ is the probability density function of Y^N on $\prod_{i=1}^N R^l$, corresponding to $M(s)$. Now suppose that the probability distribution of the observations is known. Then a reasonable criterion for selecting an approximate model for Y^N from \bar{M} is that the expected value of the log likelihood function is maximized, i.e.,

Manuscript received December 3, 1979; revised January 29, 1980 and August 29, 1980. Paper recommended by S.I. Marcus, Chairman of the Stochastic Control Committee.

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$$\max_{s \in S} \int f(Y^N) \log f_s(Y^N) dY^N = \max_{s \in S} E\{\log f_s(Y^N)\} \tag{2.2}$$

where E denotes expectation taken with respect to the known distribution of Y^n , $f(Y^N)$. As noted in [14], the integral in (2.2) may be viewed as a version of the ambiguity function used in radar applications. We call it here the probabilistic ambiguity between the actual system and the approximating model. It should be noted that in the above definition it is assumed that when y_i has a well-defined probability density on R^l (i.e., that its probability measure is either discrete or absolutely continuous with respect to the Lebesgue measure). This is suitable for the Gaussian case treated in the sequel. When this is not the case (i.e., when Y may take values from mixed continuous and discrete sets), the ambiguity function is ill-defined. Instead, the approximation criterion would be defined on Kullback-Leibler's information [15] between the corresponding probability measures, i.e.,

$$\max_{s \in S} \int \left[\log \frac{dP_s(Y^N)}{dP(Y^N)} \right] dP(Y^N)$$

assuming that the two measures are mutually absolutely continuous. It is not difficult to see that whenever these measures are absolutely continuous with respect to the Lebesgue measure, the two criteria coincide.

We shall find it convenient, without losing generality, to write the criterion in the form

$$\min_{s \in S} J(s) \tag{2.3}$$

where

$$J(s) = -\frac{2}{N} E \log f_s(Y^N).$$

In order to evaluate the compatibility of the selected approximate model, it is reasonable to compare the value of the criterion function for the selected model with the one corresponding to the given distribution. Furthermore, a good measure should be confined to a specified range, so as to provide comparative information on different approximations. Defining

$$K' = \frac{J(s) - J(*)}{J(s)}$$

it can be seen that

$$0 \leq K' \leq 1 \tag{2.5}$$

(since $E \log f_s(Y^N) \leq E \log f(Y^N) \leq \log E f(Y^N) = 0$). Clearly, K' measures the difference between the approximate model and the actual one. A measure of approximation is then defined as the complement of K' , i.e.,

$$K = 1 - K' = \frac{J(*)}{J(s)}. \tag{2.6}$$

Obviously, we have

$$0 \leq K \leq 1.$$

Of course, we could start by introducing K as the ratio in (2.6). It seems, however, that the above derivation via K' is better motivated. We call K the approximation coefficient.

2) *Linear Gaussian Systems:* Consider a linear model in state-space form

$$\begin{aligned} x_{n+1} &= F_n x_n + D_n u_n + G_n w_n \\ y_n &= H_n x_n + v_n \quad n=1, \dots, N \end{aligned} \tag{2.7}$$

where x_n , y_n , and u_n are k , l , and m -dimensional, x_0 is normally distributed with mean ξ and covariance ψ , (u_n) is a known sequence, and (w_n) and (v_n) are mutually uncorrelated, zero mean white noise sequences with covariances (Q_n) and (R_n) , respectively. We denote the given model by

$$M = \{(F_n, G_n, D_n, H_n, \xi, \psi, Q_n, R_n), n=1, \dots, N\}. \tag{2.8}$$

A time-invariant model of the form

$$\begin{aligned} x_{n+1} &= Fx_n + Du_n + Gw_n \\ y_n &= Hx_n + u_n \quad n=1, \dots, N \end{aligned} \tag{2.9}$$

that approximates M in some sense is required. This simplified model may also be of order less than k . It is reasonable, although not essential to our analysis, to restrict this simplified model to some standard (e.g., standard controllable) form. The simplified model set is denoted by

$$\bar{M} = \{M_s = (F_s, G_s, D_s, H_s, \xi_s, \psi_s, Q_s, R_s); s \in S \in R\} \tag{2.10}$$

where s is the vector of free parameters of the model and S is the parameter set. In order to write the approximation criterion for this problem we first note that

$$\log f_s(Y^N) = \sum_{n=1}^N \log f_s(y_n | Y^{n-1}) \tag{2.11}$$

where $f_s(y_n | Y^{n-1})$ denotes the conditional probability density corresponding to M_s . For the normal models concerned we have

$$\begin{aligned} f_s(y_n | Y^{n-1}) &= [(2\pi)^l |\Sigma_{s,n}|]^{-1/2} \\ &\cdot \exp \left\{ -1/2 (y_n - \hat{y}_{s,n})^T \Sigma_{s,n}^{-1} (y_n - \hat{y}_{s,n}) \right\} \end{aligned} \tag{2.12}$$

where

$$\hat{y}_{s,n} = E_s \{ y_n | Y^{n-1} \} \tag{2.13}$$

and

$$\Sigma_{s,n} = E_s \{ (y_n - \hat{y}_{s,n})(y_n - \hat{y}_{s,n})^T \}. \tag{2.14}$$

Denoting

$$J_n(s) = \log |\Sigma_{s,n}| + \text{tr} \Sigma_{s,n}^{-1} \Gamma_{s,n} + \log (2\pi)^l \tag{2.15}$$

where

$$\Gamma_{s,n} = E\{(y_n - \hat{y}_{s,n})(y_n - \hat{y}_{s,n})^T\} \quad (2.16)$$

we have

$$J(s) = \frac{1}{N} \sum_{n=1}^N J_n(s). \quad (2.17)$$

The first and the second derivatives of $J(s)$ with respect to s , required for a gradient-type minimization procedure, are provided in the Appendix.

3) *Time Invariant Systems*: In reducing the order of time invariant systems, two types of approximations may be considered in the time domain:

- a) transient response or finite time design and
- b) steady-state design.

When the system's operation is given on a finite time length, or when emphasis of the transient response characteristics is desired in the design, the approximation criterion function is given by

$$J(s) = \frac{1}{N} \sum_{n=1}^N J_n(s) \quad (2.18)$$

where N is the given (sampled) time length, the duration of the transient response, or an arbitrary initial time period selected for design purposes. Characteristic response times may be determined by eigenvalue analysis or simulation of the original system.

When an emphasis of the steady-state characteristics of the original system is required in the reduced-order design, the criterion function takes its steady state value, i.e.,

$$\begin{aligned} J(s) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N J_n(s) \\ &= \log|\Sigma_s| + \text{tr} \Sigma_s^{-1} \Gamma_s + \log(2\pi)^l \\ &= J_{s.s.}(s) \end{aligned} \quad (2.19)$$

where Σ_s and Γ_s are the steady-state values of $\Sigma_{s,n}$ and $\Gamma_{s,n}$, respectively.

B. Special Cases and Examples

We now turn to examine the usefulness of the proposed approach by solving specific low-order examples for a variety of model simplification problems. We first consider the problem of order reduction for time invariant models, using steady-state and initial response design procedures. The effect of a deterministic input on the design is also examined. We then consider the approximation of a time-varying system by a time-invariant model, and finally the approximation of a time-varying system by a lower order, time-invariant model is demonstrated.

Example 1: Given a second-order system

$$\begin{aligned} x_{n+1} &= \begin{bmatrix} 0 & 1 \\ -a & 1 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_n \\ y_n &= [1 \quad 0] x_n + v_n \quad R=Q=1 \end{aligned} \quad (2.20)$$

TABLE I
PARAMETER AND APPROXIMATION COEFFICIENT VALUES FOR
EXAMPLE 1

| Case | a | λ_1, λ_2 | s_1 | K_1 | s_2 | K_2 |
|------|--------|------------------------|-------|--------|-------|--------|
| 1 | 0.0099 | 0.01 0.99 | 0.99 | 0.9999 | 0.86 | 0.9685 |
| 2 | 0.09 | 0.90 0.10 | 0.91 | 0.9985 | 0.87 | 0.9685 |
| 3 | 0.16 | 0.80 0.20 | 0.85 | 0.9961 | 0.83 | 0.9694 |
| 4 | 0.25 | 0.50 0.50 | 0.78 | 0.9987 | 0.77 | 0.9682 |
| 5 | 0.5 | 0.5+0.5j 0.5-0.5j | 0.62 | 0.9340 | 0.64 | 0.9472 |
| 6 | 0.9 | 0.5+0.8j 0.5-0.8j | 0.50 | 0.4627 | 0.52 | 0.8331 |

s_1, K_1 - steady-state design values
 s_2, K_2 - initial response design values

it is desired to find a first-order model of the form

$$\begin{aligned} x_{n+1} &= s x_n + w_n \\ y_n &= x_n + v_n \quad R_s = Q_s = 1, \quad s \in S = (-\infty, \infty) \end{aligned} \quad (2.21)$$

that approximates the given model. Using our approximation criterion, it is necessary to find the value of s that maximizes the function $J(s)$. We seek a steady-state approximation and an initial response approximation for the first 5 sampling times. "Classical" design considerations imply that the frequency response of the approximate model should be similar to that of the original system in the design range. The optimal values of the parameter s and the corresponding approximation measure K for several values of a are given in Table I. The poles of the given system and the approximating models in the z -plane are shown in Fig. 1.

It is seen that when the original system has a dominant (low-frequency) pole (Case 1), both designs give models whose poles are close to the dominant pole. In particular, the pole of the steady-state approximation is practically identical to the low-frequency pole, while the pole corresponding to the initial response design is of higher frequency, as expected. As the poles of the original system come closer (Cases 2-4), the pole of the approximate model lies to their right, for both designs, to give similar frequency responses to that of the original poles, in the respective ranges. (Note again, that the steady-state design pole lies to the right of the initial response design pole.) Note that for Cases 1-4 the approximation coefficient assumes higher values for the steady-state design, indicating better approximation. When the poles of the original system are complex (Cases 5, 6), each of the approximate models has a real pole with a lower real frequency than the complex couple, giving similar frequency response characteristics. As the imaginary part of the original system's poles increases, the approximate model's poles converges to the real part of the original system's pole. Note that here

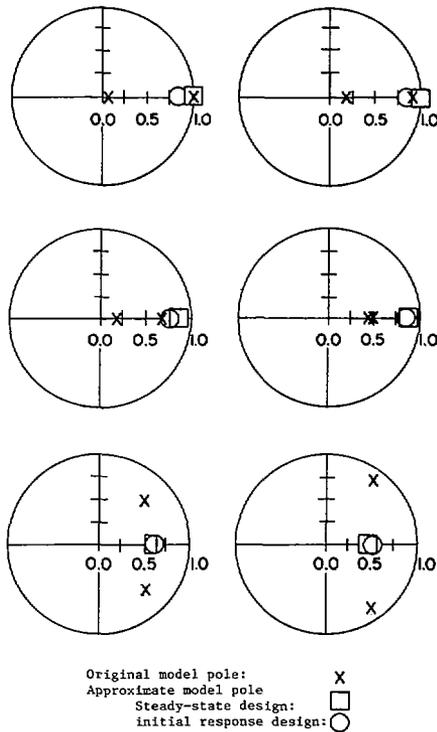


Fig. 1. z-plane poles of second-order model and first-order approximations.

the approximation coefficients for the steady-state design decreases drastically (Case 6), while its value for the initial response design remains relatively high. This means that when the original system has oscillatory characteristics (complex poles), a transient response approximation is more suitable than a steady-state approximation.

Next consider the system given by (2.20) with $a=0.25$, but let the reduced order model be

$$\begin{aligned} x_{n+1} &= s_1 x_n + w_n \\ y_n &= s_2 x_n + v_n \\ Q_s &= s_3 \quad R_s = s_4. \end{aligned}$$

A steady-state design is desired. The resulting parameter values were found to be: $s_1=0.6785$, $s_2=1.203$, $s_3=1.299$, and $s_4=0.467$ with the approximation coefficient $K=0.9993$. As might have been expected, the approximation here is better than in the previous design (Case 1), as effectively all the model's parameters were free to take their optimal values on R^4 .

Finally, consider the system given by (2.20) with an additional deterministic input, i.e.,

$$\begin{aligned} x_{n+1} &= \begin{bmatrix} 0 & 1 \\ -a & 1 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_n \\ y_n &= [1 \quad 0] x_n + v_n \quad R=Q=1. \end{aligned}$$

Let

$$u_n = 1.$$

The desired reduced-order model is of the form

$$\begin{aligned} x_{n+1} &= s x_n + u_n + w_n \\ y_n &= x_n + v_n \quad R_s = Q_s = 1, \quad s \in S = (-\infty, \infty) \end{aligned}$$

with

$$u_n = 1.$$

For $a=0.25$ and a steady-state design we get $s=0.75$ and $K=0.9856$. Comparing these results to Case 4 above, we see that the additional deterministic input has a rather marginal effect on the pole selection and on the approximation quality.

Example 2: Given a time-varying system

$$\begin{aligned} x_{n+1} &= F_n x_n + w_n \\ y_n &= x_n + v_n \\ \xi_0 &= 0 \quad \psi_0 = 1 \\ R_n &= Q_n = 1 \end{aligned}$$

consider the following cases:

a) "Fading Memory":

$$F_n = 1 - \frac{n}{10} \quad n=1, \dots, 10.$$

b) "Intensifying Memory":

$$F_n = \frac{n}{10} \quad n=0, 1, \dots, 9.$$

Time-invariant models of the form

$$\begin{aligned} x_{n+1} &= s x_n + w_n \\ y_n &= x_n + v_n \\ \xi_0 &= 0 \quad \psi_0 = 1 \\ R_{s,n} &= Q_{s,n} = 1 \end{aligned}$$

that approximate the given time-varying systems are desired. It is reasonable to expect the resulting parameter values to be close to the average (0.45 in both cases) tending, on the one hand, towards later F_n values (due to the fading past effect of a stable system) and, on the other, towards larger F_n values (which have a greater effect on the output covariance as they bring the system closer to instability). This means that the selected value of the parameter s should be greater for the intensifying memory case.

The resulting values of the parameter s and the approximation coefficient are given below.

| case | s | K |
|------|------|-------|
| a | 0.51 | 0.980 |
| b | 0.57 | 0.986 |

It is seen that in both cases average values were selected. The influence of greater F_n values is seen from the fact that in both cases $s > 0.45$. This effect is stronger for the case of intensifying memory, as expected. Also, note that the approximation coefficient is somewhat higher (i.e., the approximation is somewhat better) in the latter case, which might be explained by the fact that the system's dominant

characteristics are somewhat more definite in the latter case.

Example 3: Given the second-order time-varying system

$$\begin{aligned} x_{n+1} &= \begin{bmatrix} 0 & 1 \\ 0.1n & 1 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_n \\ y_n &= [1 \ 0] x_n + v_n \quad n=0,1,\dots,9 \\ R &= Q = 1 \end{aligned}$$

it is desired to find a first-order time-invariant model as in Example 2. The original system's eigenvalues are listed below.

| | | | | | | | | | | |
|--------------|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| <i>n</i> : | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| eigenvalues: | 1.00 | 1.09 | 1.17 | 1.24 | 1.31 | 1.37 | 1.42 | 1.47 | 1.52 | 1.57 |
| | 0.00 | -0.09 | -0.17 | -0.24 | -0.31 | -0.37 | -0.42 | -0.47 | -0.52 | -0.57 |

Employing our representation criterion we get for the approximate model $s=1.45$ with $K=0.9790$. Note that the selected s value tends towards later values of the "unstable" eigenvalue of the original system, as might have been expected. The fact that both the original system and the approximate model are unstable is of no major consequence here, since the system's operation and the approximation are required for the first 10 samples only.

III. SET REDUCTION

A. General

It is sometimes desired to reduce a model set defined by a structure and a set of parameter values to a single model with fixed parameter values and the same or simpler structure that represents the model set in some sense. In certain cases, however, important characteristics (e.g., those dynamic characteristics which are particular to subsonic and supersonic flights in modeling aircraft dynamics) may be lost in a single model design. A finite group of representative models that "spans" the model set in some sense would then be desired.

In the previous section we considered the problems of order and time structure reduction. We had a known stochastic model for the system and a model set from which a simpler approximate model was to be selected. This was achieved by minimizing a probabilistic ambiguity function with respect to a single variable parameter. For the model set reduction problem considered in the present section, the original model becomes a model set and the ambiguity function becomes a function of two variable parameters. The two parameter sets may or may not coincide, depending on whether structure reduction is also sought and on the set reduction procedure used.

B. Set Reduction Criteria

Let $S \in R^p$ and $T \in R^q$ be two parameter sets, so that for each $s \in S$ and $t \in T$ there exist probability density func-

tions $f_s(Y^N)$ and $f_t(Y^N)$ for an observation record $Y^N = (y_1, \dots, y_N)$ on its space of definition. A scalar function is defined on the product set $S \times T$ by

$$\begin{aligned} J_N(t, s) &= -\frac{2}{N} E_t \log f_s(Y^N) \\ &= -\frac{2}{N} \sum_{n=1}^N E_t \log f_s(y_n | Y^{N-1}) \\ &= -\frac{2}{N} \sum_{n=1}^N J_n(t, s) \end{aligned} \quad (3.1)$$

where E_t denotes expectation with respect to the density $f_t(Y^N)$. Note that $J_N(t, s)$ is the negative probabilistic ambiguity between the models corresponding to s and t . When the observations are ergodic (e.g., stationary Gaussian) then $J(t, s)$ has a limit value

$$J(t, s) = \lim_{N \rightarrow \infty} J_N(t, s) = -2 E_t \log f_s(Y_n | Y^{N-1}). \quad (3.2)$$

Example 4: Consider a second-order linear Gaussian model set in state-space form

$$\begin{aligned} x_{n+1} &= \begin{bmatrix} 0 & 1 \\ t & 1 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_n \\ y_n &= [1 \ 0] x_n + v_n \\ E\{w_n\} &= E\{v_n\} = 0 \\ \text{cov}\{w_n\} &= \text{cov}\{v_n\} = 1 \end{aligned}$$

where

$$-0.25 \leq t \leq 0$$

and a first-order model set

$$\begin{aligned} x_{n+1} &= s x_n + w_n \\ y_n &= x_n + v_n \\ E\{w_n\} &= E\{v_n\} = 0 \\ \text{cov}\{w_n\} &= \text{cov}\{v_n\} = 1 \end{aligned}$$

where

$$0.5 \leq s \leq 1.0.$$

The function $J(s, t)$ for this example has been plotted using a three-dimensional plotting program and is given in Fig. 2.

If Y^N does not possess well-defined densities, $J(t, s)$ will be replaced by Kullback-Leibler's information between the corresponding probability measures

$$I(t, s) = \int_{Y^N} \left[\log \frac{dP_s(Y^N)}{dP_t(Y^N)} \right] dP_t(Y^N)$$

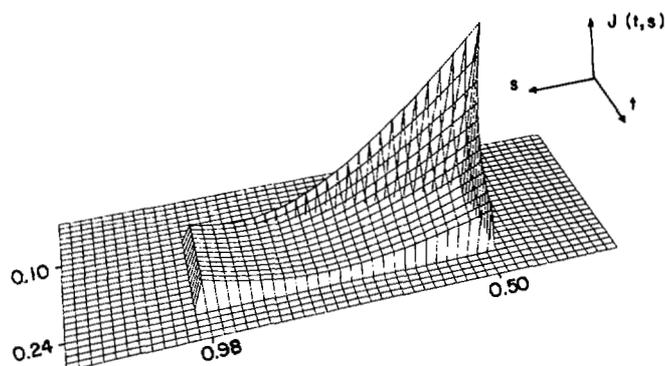


Fig. 2. The function $J(t, s)$ for Example 4.

provided that P_s and P_t are mutually absolutely continuous for all $s \in S, t \in S$. We shall see that in contrast to the structure reduction problems treated in Section II, the measures $J(t, s)$ and $I(t, s)$ do not coincide for the set reduction problem treated in this section, even when $f_s(Y)$ and $f_t(Y)$ are well defined, and that there is an obvious computational advantage to using $J(t, s)$ when possible.

Given two parameter sets T and S and two model sets defined on them, the problem of concern here is to find a finite group of models defined on S that best represents the entire model set defined on T . We call T the set of definition (of the original model) and S the set of representation. S and T may, but need not, coincide and each may be continuous or discrete. When the original model is completely specified by a parameter value t , a reasonable criterion for selecting an approximate model from a model set defined on S is that the ambiguity between the actual model and the approximate one is maximized (see Section II), i.e.,

$$\min_{s \in S} J(t, s). \tag{3.3}$$

When the actual model is not specified, but instead is given on a parameter set T , a reasonable criterion for selecting a single representative model defined on S is that the minimum possible ambiguity between the actual model (whatever it might be) and the selected one is maximized. The criterion is then

$$\min_{s \in S} \max_{t \in T} J(t, s). \tag{3.4}$$

Next, suppose that the model set defined on T is to be represented by a finite group of L models, defined on S . Denote the finite group of parameter values to be selected by

$$P = \{s_j \in S, j = 1, \dots, L\}$$

and suppose that $t \in T$ is the actual parameter value. Then, using the criterion (3.3), the best representative parameter from P is the one giving

$$\min_{s_j \in P} J(t, s_j). \tag{3.5}$$

Now P must be selected from S so that the model obtained

by (3.5) is the best representative for T . The min-max criterion (3.4) requires that this selection satisfies

$$\min_{P \in S} \max_{t \in T} \left\{ \min_{s_j \in P} J(t, s_j) \right\}. \tag{3.6}$$

We next derive a measure of compatibility for the selected model or model group by extending the measure K defined in Section II as follows. For set reduction to a single model we obtain a min-max point s^0 . Defining

$$t^0 = \arg \max_{t \in T} J(t, s^0)$$

and

$$K'(s^0) = \frac{J(t^0, s^0) - J(t^0, t^0)}{J(t^0, s^0)}$$

it can be shown (as in Section II) that

$$0 \leq K'(s^0) \leq 1.$$

$K'(s^0)$ measures the (normalized) difference between the selected parameter (s^0) and t^0 , which is "the most different" from s^0 . $K'(s^0)$ then measures the difference between the selected model and the actual one in the "worst case." A measure of approximation is then defined as the complement

$$K(s^0) = 1 - K'(s^0) = \frac{J(t^0, t^0)}{J(t^0, s^0)}. \tag{3.7}$$

For the problem of set reduction to a finite group P^0 , we similarly define

$$t^0 = \arg \max_{t \in T} \left\{ \min_{s \in P^0} J(t, s) \right\}$$

and

$$J(t^0, P^0) = \max_{t \in T} \left\{ \min_{s \in P^0} J(t, s) \right\}.$$

Then a measure of difference between the selected model and the actual one for the "worst case" is

$$K'(P^0) = \frac{J(t^0, P^0) - J(t^0, t^0)}{J(t^0, P^0)}$$

and a measure of approximation is defined by the complement

$$K(P^0) = 1 - K'(P^0) = \frac{J(t^0, t^0)}{J(t^0, P^0)}. \tag{3.8}$$

C. The Optimization Problem

The problem at hand is to construct procedures for solving the optimization problems (3.4) and (3.6). Min-max problems have received considerable attention in the litera-

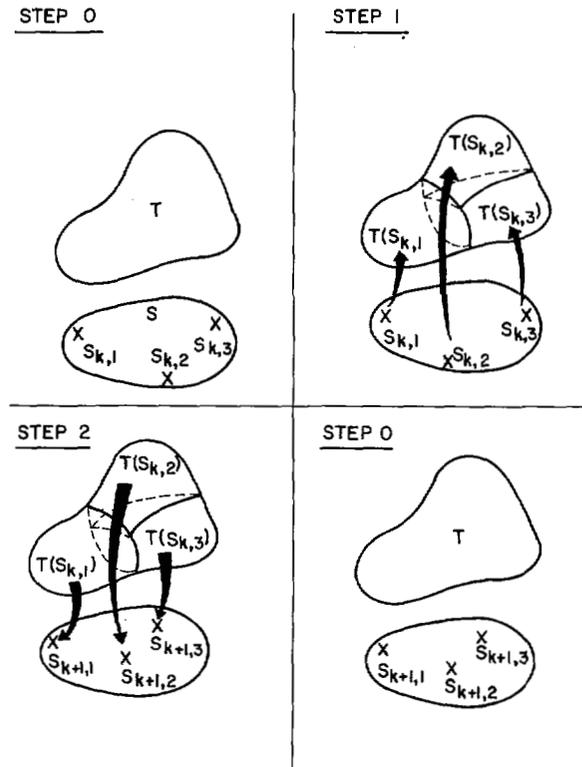


Fig. 3. Iterative reduction of a continuous set to a finite group.

ture and several procedures can be adapted to (3.4). On the other hand, there does not seem to be anything in the literature concerning problems such as (3.6), which is a generalization of the min-max problem. The following procedure reduces the solution of (3.6) into iterative solutions of min-max problems under constraints.

Step 0: On the k th iteration, have the group

$$P_k = (s_{k,j}; j=1, \dots, L).$$

Step 1: For each $s_{k,j} \in P_k$ find a subset $T(s_{k,j}) \in T$ such that for $t \in T(s_{k,j})$

$$J(t, s_{k,j}) \geq J(t, s_{k,i}) \quad \text{for any } i \neq j.$$

Step 2: Find the min-max point in S for each $T(s_{k,j})$, $j=1, \dots, L$, and obtain a new group $P_{k+1} = (s_{k+1,j}, j=1, \dots, L)$.

Step 3: Check a stopping condition. Set $k=k+1$ and repeat Steps 1 and 2 until stopping condition is satisfied.

The above procedure is described in Fig. 3. The actual algorithm performs Steps 1 and 2 together as follows.

Step 1, 2: For each $s_{k,j} \in P_k, j=1, \dots, L$

find

$$\min_{s \in S} \max_{t \in T} J(t, s) \tag{3.9}$$

subject to

$$J(t, s_{k,j}) \geq J(t, s_{i,j}) \text{ for any } i \neq j. \tag{3.10}$$

Note that the constraint involves only the parameter t and not s . It follows that in solving (3.9) in each iteration only the maximization is performed under the constraint (3.10).

The solution of (3.4) is then a (considerably simpler) subproblem of the solution of (3.6). This should come as no surprise as representation by a single model is a special case of representation by a finite group. The representation problems at hand involve solving unconstrained (in the case of a single representative model) or constrained (in the case of a representative group) min-max problems. These, in turn, involve unconstrained minimization and constrained or unconstrained maximization steps. Note that for the case of linear Gaussian models on continuous parameter sets exact expressions for the first and the second derivatives of the criterion function $J(s, t)$ with respect to the parameter s are provided in the Appendix, and can be used in performing the minimization step on S . The derivatives with respect to the parameter t for the maximization step on T can be derived in a similar manner (replacing the original model by the model corresponding to t and differentiating with respect to t).

D. Special Cases

1) General: Before addressing the optimization problems (3.4) and (3.6) for continuous parameter sets S and T , we consider the case where both sets are discrete, and the case where one set is continuous and the other discrete. These cases may be viewed as valid optimization problems in their own right, applying to practical problems of selecting between finite alternatives, or as approximations of the continuous parameter problem. Such simplified versions of the optimization problem are also used as intermediate iterative steps in certain algorithms for solving min-max problems on continuous parameter sets (e.g., 16). We shall

use (a, b) to denote the finite set composed of a and b and $[a, b]$ to denote the closed interval between a and b .

2) *Finite Parameter Sets*: The parameter set T may be given as a finite set or may be constructed as such from a continuous set by some discretization, so as to approximate the original set. The set of representation S may coincide with T or may define a finite set of simpler models (say, of lower order) than those defined by T . The solution of the min-max problem (3.4) is then found by simple comparison of $J(s, t)$ values of pairs s and t . The solution of (3.6) is found by the procedure proposed in the previous section, solving the min-max problem on each iteration.

Example 5: Given the model set

$$\begin{aligned} x_{n+1} &= tx_n + w_n & t \in T \\ y_n &= x_n + v_n \\ x_0 &= 0 \quad \Psi_0 = 1 \\ R &= Q = 1 \end{aligned}$$

it is desired to find a single representative model from the set for the following cases:

- a) steady-state design for $T=(0.1, 0.2, 0.3, 0.4, 0.5, 0.6)$,
- b) steady-state design for $T=(0.1, 0.2, 0.3, 0.8, 0.9, 0.99)$.
- c) initial response design (first 3 samples) for $T=(0.1, 0.2, 0.3, 0.8, 0.9, 0.99)$.

The representative models are found by applying the criterion (3.4) with $S=T$. The selected values of t^0 and the approximation measure $K(t^0)$ (which are given for comparison with later results) are listed below.

| Case | t^0 | $K(t^0)$ |
|------|-------|----------|
| a | 0.4 | 0.9836 |
| b | 0.99 | 0.8223 |
| c | 0.8 | 0.8945 |

It is seen that in Case *a* an "average" value for the parameter was selected from the set. In Case *b*, however, the extremal value was selected. This is explained by the fact that the system associated with 0.99 is nearly unstable and the resulting higher output covariance has greater influence on the criterion function $J(t, s)$ than the other models. This is a desirable result, as the nearly unstable models should be approximated more closely than the strongly stable ones. For the initial response design, Case *c*, the selected parameter value has shifted to the left, as the effect of near instability is small for the short time period.

Example 6: Now it is desired to find the best two representative models from the model set of Example 4 with $T=(0.1, 0.2, 0.3, 0.4, 0.5, 0.6)$ for a steady-state design. Starting with the initial parameters $P_0=(0.2, 0.4)$ the constrained min-max problem is, on the first iteration,

$$\min_{s \in T} \max_{t \in T} J(s, t)$$

subject to (maximization step only)

$$J(t, 0.2) \leq J(t, 0.4)$$

and

$$\min_{s \in T} \max_{t \in T} J(s, t)$$

subject to

$$J(t, 0.4) \leq J(t, 0.2).$$

After computing $J(t, 0.2)$ and $J(t, 0.4)$ values for all $t \in T$, the above constrained min-max problems become

$$\min_{s \in T_1} \max_{t \in T_1} J(s, t) \quad T_1 = (0.1, 0.2, 0.3)$$

and

$$\min_{s \in T_2} \max_{t \in T_2} J(s, t) \quad T_2 = (0.4, 0.5, 0.6)$$

yielding

$$P_1 = (0.2, 0.5)$$

which remains unchanged in further iterations.

3) *Finite-Continuous Parameter Sets*: Intermediate approximations for continuous set reduction may be achieved by discretizing one of the parameter sets, yielding one of the following two situations:

- a) discrete definition set and continuous representation set,
- b) continuous definition set and discrete representation set.

Either discretization would reduce the computation effort involved in solving the continuous set reduction problem considerably, as the maximization step (Case *a*) or the minimization step (Case *b*) of the min-max procedure are performed on finite sets. There does not seem to be an obvious preference to *a* or *b* a priori, unless they are suggested by the given problem (for instance, Case *a* would arise in finite set reduction when structure reduction is also performed).

Optimization procedures for solving min-max problems when one parameter is given on a continuous set and the other on a discrete finite set have been suggested in the literature [17] and some are available in widely used subroutine libraries [18]. These subroutines may be used to solve our min-max problem (3.4). Note, however, that these general subroutines are tailored for problems where only the criterion function (and none of its derivatives) is available or where only the function and its first derivative are available. In our case we have first and second derivatives available for linear Gaussian models, so that better solutions (faster convergence) may be achieved.

Example 7: It is desired to find a single representative from the set

$$\begin{aligned} x_{n-1} &= sx_n + w_n \\ y_n &= x_n + v_n \\ x_0 &= 0, \quad \Psi_0 = 1 \\ R &= Q = 1 \end{aligned}$$

where $s \in S = [0.1, 0.6]$, the closed interval between 0.1 and 0.6. The set S is first "approximated" by a finite set $T = (0.1, 0.2, 0.3, 0.4, 0.5, 0.6)$. Performing the optimization

$$\min_{s \in S} \max_{t \in T} J(t, s)$$

for a steady-state design, we obtain for the representative model $s^0 = 0.406$ and $K(s^0) = 0.9840$. Comparing these values to the ones obtained in Example 5 (Case *a*) we see that the selected values are very close, while the representation here is slightly better (larger K^0) than in the finite-finite case, as might have been expected.

Example 8: Let a discrete model set be given by

$$\begin{aligned} x_{n+1} &= \begin{bmatrix} 0 & 1 \\ t & 1 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_n \\ y_n &= [1 \quad 0] x_n + v_n \quad R=Q=1 \end{aligned}$$

where

$$t \in T = (-0.09, -0.16, -0.25).$$

The eigenvalues of the system matrix are listed below.

| t | eigenvalues |
|-------|-------------|
| -0.09 | 0.9, 0.1 |
| -0.16 | 0.8, 0.2 |
| -0.25 | 0.5, 0.5 |

It is desired to find a first order representative model for the above set, i.e.,

$$\begin{aligned} x_{n+1} &= s x_n + w_n \\ y_n &= x_n + v_n \quad R=Q=1 \end{aligned}$$

where s may take any value on the real line. Applying the min-max criterion we get $s^0 = 0.842$. Note that the selected value is close to the higher eigenvalues of the original models, which is explained by the higher covariance values associated with the corresponding modes. From a frequency response viewpoint the approximate representative model tends towards the low-frequency range of the original system due to the steady-state design.

4) *Continuous Parameter Sets:* A simple-minded procedure for solving min-max problems on continuous parameter sets is alternating minimization and maximization steps. Convergence can be normally guaranteed when the function has a saddle point [19]. It can be argued, however, that for the function $J(s, t)$ a saddle point will not normally exist. (Note that even in the very simple case of scalar parameters, depicted in Fig. 2, a saddle point does not exist.) A procedure for solving min-max problems on continuous parameter sets by iterations on continuous-discrete sets has been proposed by Salmon [16], who also showed convergence of the algorithm, even in the absence of a saddle point. This optimization method has been used in the following example.

Example 9: Let a first-order linear system be given by

$$\begin{aligned} x_{n+1} &= t x_n + w_n \\ y_n &= x_n + v_n \quad R_t = Q_t = 1 \end{aligned}$$

where t takes its values from a given interval T on the real line. It is desired to find a single model that best represents the model set in steady-state. Applying our representation criterion (3.4) for two cases of T , the following values for t^0 were obtained.

| Case | T | t^0 |
|----------|------------|-------|
| <i>a</i> | [0.2, 0.5] | 0.370 |
| <i>b</i> | [0.1, 0.6] | 0.406 |

Note that the selected value of t^0 is greater than the mean value of the interval, which is explained by the fact that models with higher t values produce greater values of the output covariance (are "closer" to instability) and have greater influence on the criterion function. This effect is emphasized by the fact that although in Case *b* we have a symmetrical extension of the interval of Case *a*, the selected value of t^0 has shifted to the right.

Example 10: We now wish to find the best two and the best three representative models for the continuous-parameter model set of Example 9 with $T = [0.1, 0.6]$. The representation criterion is now (3.6). The selected parameter values are listed below along with the result obtained for a single representative model and the approximation measure, for comparison

| | | |
|------------------------|--------------------|------------------|
| one representative: | (0.406); | $K(s^0) = 0.984$ |
| two representatives: | (0.25, 0.50); | $K(P^0) = 0.996$ |
| three representatives: | (0.2, 0.39, 0.54); | $K(P^0) = 0.998$ |

Note again, that the representations tend to emphasize the higher range of the parameter set for reasons discussed before. Also note that as should be expected, the min-max approximation improves (higher K values) when more representative models are taken.

IV. CONCLUSION

The problem of stochastic model simplification has been treated in the contexts of structure reduction and parameter set reduction. The approximation of high-order and time-varying systems by low-order and time-invariant models is achieved by maximizing the probabilistic ambiguity between the actual system and the selected model. The reduction of a model set defined on a set of parameter variation or uncertainty into a single model or a finite model group is achieved by finding the min-max and the min-max-min points of the ambiguity function. Set reduction may include structure reduction as a subproblem. The algorithmic aspects of the model simplification problem were considered in some detail. It should be emphasized that any efficient, general minimization, min-max and constrained min-max algorithms may be used to solve the respective model simplification problems. Several such algorithms have been tested and have shown good convergence properties for the examples considered. For linear Gaussian models exact expressions for the gradient and the Hessian are computable, so that better convergence rates can be achieved at the cost of more extensive computation per iteration. An efficient optimization procedure may employ a general algorithm, approximating the first or the second derivative, as a first stage, and a specialized algorithm, using the exact derivatives, as a second stage, for higher sensitivity in the neighborhood of the solution.

Several discretization and approximation schemes have been considered. The validity of the proposed theory and the applicability of the optimization techniques have been demonstrated by a variety of numerical examples, concerning linear-Gaussian models. The resulting designs have been shown to agree with the intuitive choice of "average" or "typical" models and with traditional system theoretic considerations.

APPENDIX

The basic iteration in a gradient-type nonlinear programming method is

$$s^{k+1} = s^k - \rho_k H(s^k) \left. \frac{\partial J(s)}{\partial s} \right|_{s=s^k} \quad (\text{A.1})$$

where s^k is the parameter vector at the k th iteration, $H(s^k)$ is the inverse of the Hessian—the second partial matrix ($\partial^2 J(s)/\partial s^k|_{s=s^k}$), and ρ_k is the scalar step size parameter.

In order to apply the method to our problem, the gradient and Hessian of $J(s)$ must be derived. Clearly, these are obtained by summation of the gradients and Hessians of $J_n(s)$, $n=1, \dots, N$, which are derived in the sequel.

A. Gradient Computation

From (2.15)

$$J_n(s) = \log|\Sigma_{s,n}| + \text{tr} \Sigma_{s,n}^{-1} \Gamma_{s,n} \quad (\text{A.2})$$

using rules of matrix differentiation, we get

$$\begin{aligned} \frac{\partial J_n(s)}{\partial s_i} = & \text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial \Sigma_{s,n}}{\partial s_i} \right] + \text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial \Gamma_{s,n}}{\partial s_i} \right] \\ & - \text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial \Sigma_{s,n}}{\partial s_i} \Sigma_{s,n}^{-1} \Gamma_{s,n} \right] \quad (\text{A.3}) \end{aligned}$$

where s_i is the i th term of the parameter vector s . $\Sigma_{s,n}$ is obtained from

$$\Sigma_{s,n} = H_s \psi_{s,n} H_s^T + R_s \quad (\text{A.4})$$

where $\psi_{s,n}$ is computed from the Riccati equation

$$\psi_{s,n+1} = F_s \psi_{s,n} F_s^T + G_s Q_s G_s^T - F_s K_{s,n} \Sigma_{s,n} K_{s,n}^T F_s^T \quad (\text{A.5})$$

initialized at $\psi_{s,0} = \psi_s$, where $K_{s,n} = \psi_{s,n} H_s^T [H_s \psi_{s,n} H_s^T + R_s]^{-1}$ is the Kalman gain matrix.

The gradient of $\Sigma_{s,n}$ is obtained from (A.4) as

$$\frac{\partial \Sigma_{s,n}}{\partial s_i} = H_s \frac{\partial \psi_{s,n}}{\partial s_i} H_s^T + \frac{\partial R_s}{\partial s_i} + \frac{\partial H_s}{\partial s_i} \psi_{s,n} H_s^T + H_s \psi_{s,n} \frac{\partial H_s^T}{\partial s_i} \quad (\text{A.6})$$

and using (A.5) we have

$$\frac{\partial \psi_{s,n+1}}{\partial s_i} = \bar{F}_{s,n} \frac{\partial \psi_{s,n}}{\partial s_i} \bar{F}_{s,n}^T + \Omega_{s_i} + \Omega_{s_i}^T \quad (\text{A.7})$$

where

$$\bar{F}_{s,n} \equiv F_s [I - K_{s,n} H_s]$$

and

$$\begin{aligned} \Omega_{s_i} \equiv & \frac{\partial F_s}{\partial s_i} \psi_{s,n} \bar{F}_{s,n}^T - F_s K_{s,n} \frac{\partial H_s}{\partial s_i} \psi_{s,n} \bar{F}_{s,n}^T \\ & + \frac{1}{2} \frac{\partial}{\partial s_i} [G_s Q_s G_s^T] + \frac{1}{2} F_s K_{s,n} \frac{\partial R_s}{\partial s_i} K_{s,n}^T F_s^T. \end{aligned}$$

$\Gamma_{s,n}$ is obtained from

$$\Gamma_{s,n} = \bar{H}_s [\bar{\psi}_{s,n} + \bar{x}_{s,n} \bar{x}_{s,n}^T] \bar{H}_s^T \quad (\text{A.8})$$

where $\bar{\psi}_{s,n}$ is obtained recursively from the equation

$$\bar{\psi}_{s,n+1} = \bar{F}_{s,n} \bar{\psi}_{s,n} \bar{F}_{s,n}^T + \bar{G}_{s,n} \bar{Q} \bar{G}_{s,n}^T \quad (\text{A.9})$$

initialized at

$$\bar{\psi}_{s,0} = \begin{bmatrix} \psi_s & 0 \\ 0 & 0 \end{bmatrix}$$

and $\bar{x}_{s,n}$ is obtained from

$$\bar{x}_{s,n+1} = \bar{F}_{s,n} \bar{x}_{s,n} + \bar{D}_s u_n \quad (\text{A.10})$$

initialized at

$$\bar{x}_{s,0} = 0.$$

We have used the notation

$$\begin{aligned} \bar{F}_{s,n} = & \begin{bmatrix} F & 0 \\ F_s K_{s,n} H & F_s (I - K_{s,n} H_s) \end{bmatrix}; \quad \bar{G}_{s,n} = \begin{bmatrix} G & 0 \\ 0 & F_s K_{s,n} \end{bmatrix} \\ \bar{Q} = & \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}; \quad \bar{H} = \begin{bmatrix} H & -H_s \end{bmatrix} \end{aligned}$$

$$\bar{x}_{s,n} \equiv E \begin{bmatrix} x_n \\ \hat{x}_{s,n} \end{bmatrix}; \quad \bar{D}_s = \begin{bmatrix} D \\ D_s \end{bmatrix}.$$

Denoting

$$\tilde{\psi}_{s,n} = \bar{\psi}_{s,n} + \bar{x}_{s,n} \bar{x}_{s,n}^T$$

the gradient of $\Gamma_{s,n}$ is

$$\frac{\partial \Gamma_{s,n}}{\partial s_i} = \bar{H}_s \frac{\partial \tilde{\psi}_{s,n}}{\partial s_i} \bar{H}_s^T + \frac{\partial \bar{H}_s}{\partial s_i} \tilde{\psi}_{s,n} \bar{H}_s^T + \bar{H}_s \tilde{\psi}_{s,n} \frac{\partial \bar{H}_s^T}{\partial s_i} \quad (\text{A.11})$$

where

$$\frac{\partial \tilde{\psi}_{s,n}}{\partial s_i} = \frac{\partial \bar{\psi}_{s,n}}{\partial s_i} + \frac{\partial \bar{x}_{s,n}}{\partial s_i} \bar{x}_{s,n}^T + \bar{x}_{s,n} \frac{\partial \bar{x}_{s,n}^T}{\partial s_i}$$

where from (A.9)

$$\begin{aligned} \frac{\partial \bar{\psi}_{s,n+1}}{\partial s_i} &= \bar{F}_{s,n} \frac{\partial \bar{\psi}_{s,n}}{\partial s_i} \bar{F}_{s,n}^T + \frac{\partial}{\partial s_i} [\bar{G}_{s,n} \bar{Q} \bar{G}_{s,n}^T] \\ &+ \frac{\partial \bar{F}_{s,n}}{\partial s_i} \bar{\psi}_{s,n} \bar{F}_{s,n}^T + \bar{F}_{s,n} \bar{\psi}_{s,n} \frac{\partial \bar{F}_{s,n}^T}{\partial s_i} \end{aligned} \quad (\text{A.12})$$

and from (A.10)

$$\frac{\partial \bar{x}_{s,n+1}}{\partial s_i} = \bar{F}_{s,n} \frac{\partial \bar{x}_{s,n}}{\partial s_i} + \frac{\partial \bar{F}_{s,n}}{\partial s_i} \bar{x}_{s,n} + \frac{\partial \bar{D}_s}{\partial s_i} u_n. \quad (\text{A.13})$$

B. Hessian Computation

From (A.3) we get

$$\begin{aligned} \frac{\partial^2 J_n(s)}{\partial s_j \partial s_i} &= -\text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial \Sigma_{s,n}}{\partial s_j} \Sigma_{s,n}^{-1} \frac{\partial \Sigma_{s,n}}{\partial s_i} \right] + \text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial^2 \Sigma_{s,n}}{\partial s_j \partial s_i} \right] \\ &- \text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial \Sigma_{s,n}}{\partial s_j} \Sigma_{s,n}^{-1} \frac{\partial \Gamma_{s,n}}{\partial s_i} \right] + \text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial^2 \Gamma_{s,n}}{\partial s_j \partial s_i} \right] \\ &+ \text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial \Sigma_{s,n}}{\partial s_j} \Sigma_{s,n}^{-1} \frac{\partial \Sigma_{s,n}}{\partial s_i} \Sigma_{s,n}^{-1} \Gamma_{s,n} \right] \\ &- \text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial^2 \Sigma_{s,n}}{\partial s_j \partial s_i} \Sigma_{s,n}^{-1} \Gamma_{s,n} \right] \\ &+ \text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial \Sigma_{s,n}}{\partial s_i} \Sigma_{s,n}^{-1} \frac{\partial \Sigma_{s,n}}{\partial s_j} \Sigma_{s,n}^{-1} \Gamma_{s,n} \right] \\ &- \text{tr} \left[\Sigma_{s,n}^{-1} \frac{\partial \Sigma_{s,n}}{\partial s_i} \Sigma_{s,n}^{-1} \frac{\partial \Gamma_{s,n}}{\partial s_j} \right]. \end{aligned} \quad (\text{A.14})$$

For the calculation of (A.12) we need the second derivatives of $\Sigma_{s,n}$ and $\Gamma_{s,n}$. From (A.6) we derive

$$\frac{\partial^2 \Sigma_{s,n}}{\partial s_j \partial s_i} = H_s \frac{\partial^2 \psi_{s,n}}{\partial s_j \partial s_i} H_s^T + \Pi_{s,ij} + \Pi_{s,ij}^T \quad (\text{A.15})$$

where

$$\Pi_{s,ij} \equiv \frac{\partial H_s}{\partial s_j} \frac{\partial \psi_{s,n}}{\partial s_i} H_s^T + \frac{\partial H_s}{\partial s_i} \frac{\partial \psi_{s,n}}{\partial s_j} H_s^T + \frac{\partial H_s}{\partial s_j} \psi_{s,n} \frac{\partial H_s^T}{\partial s_i}$$

and from (A.7)

$$\frac{\partial^2 \psi_{s,n+1}}{\partial s_j \partial s_i} = \tilde{F}_{s,n} \frac{\partial^2 \psi_{s,n}}{\partial s_j \partial s_i} \tilde{F}_{s,n}^T + \tilde{\Omega}_{s,ij} + \tilde{\Omega}_{s,ij}^T \quad (\text{A.16})$$

where

$$\tilde{\Omega}_{s,ij} \equiv \frac{\partial \tilde{F}_{s,n}}{\partial s_j} \frac{\partial \psi_{s,n}}{\partial s_i} \tilde{F}_{s,n}^T + \frac{\partial}{\partial s_j} \Omega_{s,i}$$

From (A.11) we obtain

$$\frac{\partial^2 \Gamma_{s,n}}{\partial s_j \partial s_i} = \bar{H}_s \frac{\partial^2 \bar{\psi}_{s,n}}{\partial s_j \partial s_i} \bar{H}_s^T + \nabla_{s,ij} + \nabla_{s,ij}^T \quad (\text{A.17})$$

where

$$\nabla_{s,ij} \equiv \frac{\partial \bar{H}_s}{\partial s_j} \frac{\partial \bar{\psi}_{s,n}}{\partial s_i} \bar{H}_s^T + \frac{\partial \bar{H}_s}{\partial s_i} \frac{\partial \bar{\psi}_{s,n}}{\partial s_j} \bar{H}_s^T + \frac{\partial \bar{H}_s}{\partial s_i} \bar{\psi}_{s,n} \frac{\partial \bar{H}_s^T}{\partial s_j}$$

and from (A.9)

$$\begin{aligned} \frac{\partial^2 \bar{\psi}_{s,n+1}}{\partial s_j \partial s_i} &= \bar{F}_{s,n} \frac{\partial^2 \bar{\psi}_{s,n}}{\partial s_j \partial s_i} \bar{F}_{s,n}^T + \frac{\partial^2}{\partial s_j \partial s_i} \\ &\cdot [\bar{G}_{s,n} \bar{Q} \bar{G}_{s,n}^T] + \Delta_{s,ij} + \Delta_{s,ij}^T \end{aligned} \quad (\text{A.18})$$

where

$$\begin{aligned} \Delta_{s,ij} &\equiv \frac{\partial \bar{F}_{s,n}}{\partial s_j} \frac{\partial \bar{\psi}_{s,n}}{\partial s_i} \bar{F}_{s,n}^T + \frac{\partial \bar{F}_{s,n}}{\partial s_i} \frac{\partial \bar{\psi}_{s,n}}{\partial s_j} \bar{F}_{s,n}^T \\ &+ \frac{\partial \bar{F}_{s,n}}{\partial s_j} \bar{\psi}_{s,n} \frac{\partial \bar{F}_{s,n}^T}{\partial s_i} + \frac{\partial^2 \bar{F}_{s,n}}{\partial s_j \partial s_i} \bar{\psi}_{s,n} \bar{F}_{s,n}^T. \end{aligned}$$

From (A.13)

$$\begin{aligned} \frac{\partial^2 \bar{x}_{s,n+1}}{\partial s_j \partial s_i} &= \bar{F}_{s,n} \frac{\partial^2 \bar{x}_{s,n}}{\partial s_j \partial s_i} + \frac{\partial \bar{F}_{s,n}}{\partial s_j} \frac{\partial \bar{x}_{s,n}}{\partial s_i} + \frac{\partial \bar{F}_{s,n}}{\partial s_i} \frac{\partial \bar{x}_{s,n}}{\partial s_j} + \frac{\partial^2 \bar{F}_{s,n}}{\partial s_j \partial s_i} \bar{x}_{s,n} \end{aligned}$$

where

$$\frac{\partial^2 \bar{\psi}_{s,n}}{\partial s_j \partial s_i} = \frac{\partial^2 \bar{\psi}_{s,n}}{\partial s_j \partial s_i} + \theta_{s,ij} + \theta_{s,ij}^T$$

and

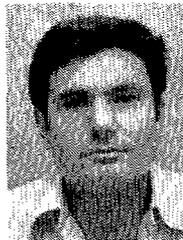
$$\theta_{s,ij} \equiv \frac{\partial^2 \bar{x}_{s,n}}{\partial s_j \partial s_i} \bar{x}_{s,n}^T + \frac{\partial \bar{x}_{s,n}}{\partial s_j} \frac{\partial \bar{x}_{s,n}^T}{\partial s_i}$$

Note that some of the equations involved are commonly used in linear filtering and covariance analysis [20] and in maximum likelihood identification [21]–[23]. Also note that the computation effort per step may be reduced (at the possible cost of reduced convergence rate) if instead of the exact Hessian matrix derived above, approximations are used. For instance, a Hessian approximation formula which employs the gradient values has been reported to perform quite effectively [24].

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