Lower Bounds on the State Complexity of Linear Tail-Biting Trellises

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***Abstract*—Lower bounds on the state complexity of linear tail-biting trellises are presented. One bound generalizes the total-span bound, while another bound can be regarded as a generalization of the cut-set bound. It is shown by examples that the new bounds may be tighter than any of the existing lower bounds.**

***Index Terms*—Linear tail-biting trellises, state complexity.**

I. Introduction

Trellis diagrams are used mainly for efficient soft-decision decoding of codes. *Conventional* trellises, defined on a linear time axis, are usually used for efficient optimum (maximum-likelihood or maximum *a posteriori*) decoding. On the contrary, *tail-biting* trellises, defined on a circular time axis, are usually used for even more efficient, although suboptimum, decoding. While the theory of conventional trellises of block codes is by now well-established (see [15] for a comprehensive study on the subject), the theory of tail-biting trellises is still in its infancy.

Tail-biting trellises were originally introduced by Solomon and van Tilborg [14]. Recently, there has been an interest in the representation of block codes by tail-biting trellises [2], [6]–[8], [11]–[13]. The renewed interest in the subject, owing probably to the recent popularity of the subject of *codes on graphs* (see, e.g., [4], [17]), started with a work of Calderbank *et al.* [2], in which efficient constructions of tail-biting trellises for several short codes were devised. The fundamental theorems for a more general theory of *linear* tail-biting trellises were derived by Kötter and Vardy [6]–[8].

As opposed to the case of conventional trellises, currently there is no feasible method to calculate the minimum possible *state complexity* of a tail-biting trellis representing a given linear code C, even when considering only linear tail-biting trellises and fixing the coordinate ordering of C. Although Kötter and Vardy [8] have devised an algorithm that, given a linear code C, produces a linear tail-biting trellis T for C with the minimum possible state complexity, the time complexity of this algorithm is not known in general. It is therefore of interest to find lower bounds on the minimum possible state complexity of a linear tail-biting trellis representing a given linear code under both its fixed coordinate ordering and an arbitrary coordinate ordering.

Several papers considered lower bounds on the state complexity of tail-biting trellises. The *square-root bound* [2], [17], an immediate consequence of the more general *cut-set bound*, indicates that the state complexity of a (linear or nonlinear) tail-biting trellis representing a linear code C is not smaller than half the state complexity of the *minimal trellis* [15] of C. A necessary condition for achieving the square-root bound for linear trellises was presented in [13]. Bounds on the state complexity of linear tail-biting trellises representing cyclic codes (under a cyclic coordinate ordering) were presented by Reuven and Be'ery [11] and by Shankar *et al.* [12]. The *total-span bound* [2] on the state complexity of a linear tail-biting trellis representing a linear code C applies to any coordinate ordering of C. A significantly tighter bound, applying also to any coordinate ordering of C, was recently introduced by Bocharova *et al.* [1].

In this correspondence, new lower bounds on the state complexity of linear tail-biting trellises are presented. Generalizing the total-span bound, a new bound on the state complexity of a linear tail-biting trellis representing a given code under an arbitrary coordinate ordering is obtained. The new bound is never looser than the bound of Bocharova *et al.* [1], and it is illustrated by an example that the new bound may be tighter than that of [1]. Moreover, the new bound also applies to the average of the *state complexity profile*, and not just to the state complexity itself. Another bound, which can be regarded as a generalization of the cut-set bound, is also derived. This *generalized cut-set bound* has both a version for a fixed coordinate ordering and a version for an arbitrary coordinate ordering. It is illustrated by examples that the bound may be tighter than any of the existing bounds.

The correspondence is organized as follows. Section II contains notation and definitions used in the subsequent sections. The *generalized total-span bound* is then presented and proved in Section III. Finally, the generalized cut-set bound is established in Section IV.

II. Preliminaries

This section contains definitions and notation used throughout the correspondence. An *edge-labeled directed graph* is a triple (V, E, A), where V is a set of vertices, A is a finite set referred to as the *alphabet*, and E is a set of labeled edges, i.e., a set of ordered triples (v, v', α) with $v, v' \in V$ and $\alpha \in A$. An edge-labeled directed graph is called *finite* whenever $|V|$ is finite, where $|V|$ is the cardinality of V. A conventional trellis $T = (V, E, A)$ of length n is a finite edge-labeled directed graph for which it is possible to write V as a union of disjoint subsets $V = V_0 \cup V_1 \cup \cdots \cup V_n$ with $|V_0| = |V_n| = 1$, such that $(v, v', \alpha) \in E$ implies that $v \in V_i$, $v' \in V_{i+1}$ for some $i \in \{0, 1, \ldots, n-1\}$. For $i \in \{0, 1, \ldots, n\}$, the subset V_i in the above partition of V is referred to as the *vertex class* of T at index i. Let $q := |A|$ be the number of elements in A. For $i \in \{0, 1, \ldots, n\}$, the number $s_i(T) := \log_q(|V_i|)$ is referred to as the state complexity of T at index i. The sequence $\boldsymbol{s}(T) := (s_0(T), s_1(T), \ldots, s_n(T))$ is called the state complexity profile of T, and its maximum value $s(T) := \max\{s_i(T) | i \in \{0, 1, \ldots, n\}\}$ is called the state complexity of T.

Let $T = (V, E, A)$ be a conventional trellis of length n. To each path

$$\mathcal{P} = v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \cdots \xrightarrow{c_{n-2}} v_{n-1} \xrightarrow{c_{n-1}} v_n$$

of T (with $v_i \in V_i$, $i \in \{0, 1, \ldots, n\}$, and $c_j \in A$, $j \in \{0, 1, \ldots, n-1\}$) we can relate the vector $\boldsymbol{c}(\mathcal{P}) := (c_0, c_1, \ldots, c_{n-1}) \in A^n$. In this way, we can relate to T a subset $C(T)$ of A^n defined as $\bigcup\{\boldsymbol{c}(\mathcal{P})\}$, where the union is over all paths from the single vertex in V_0 to the single vertex in V_n.

A tail-biting trellis $T = (V, E, A)$ of length n is a finite edge-labeled directed graph for which it is possible to write V as a union of disjoint subsets $V = V_0 \cup V_1 \cup \cdots \cup V_{n-1}$ such that $(v, v', \alpha) \in E$ implies that $v \in V_{\bar{i}}$, $v' \in V_{\overline{i+1}}$ for some integer i, where $\bar{i} := i \bmod n$. As before, the subset $V_i \subset V$ is referred to as the vertex class of T at index i, $i \in \{0, 1, \ldots, n-1\}$. The state complexity profile of T, $\boldsymbol{s}(T) := (s_0(T), s_1(T), \ldots, s_{n-1}(T))$ and the state complexity of T, $s(T) := \max\{s_i(T) | i \in \{0, 1, \ldots, n-1\}\}$ are now defined in a similar way to the definition for conventional trellises.

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A *valid path* (also called *reduction* [6], or *cycle* [7]) in a length-n tail-biting trellis $T = (V, E, A)$ is a length-n path that begins and ends at the same vertex of V_0, that is, a path of the form

$$v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \cdots \xrightarrow{c_{n-2}} v_{n-1} \xrightarrow{c_{n-1}} v_0$$

with $v_i \in V_i$, $c_i \in A$, $i \in \{0, 1, \ldots, n-1\}$. If every vertex and every edge of T lies on some valid path of T, then we say that T is *reduced* [6], [7]. Throughout this correspondence, it is assumed that tail-biting trellises are reduced. A subset $C(T)$ of A^n can be assigned to the tail-biting trellis T in a similar way as before (in the case of conventional trellises), by reading off the vectors related to all valid paths in T.

Note that in a length-n tail-biting trellis $T=(V,E,A)$ with $|V_0|=1$ (say, $V_0 = \{v_0\}$) every length-n path beginning in v_0 is a valid path. In this case, T can be regarded as a conventional trellis. Strictly speaking, it is required to introduce an additional vertex v_n to V, and to replace each edge of the form (v', v_0, α), $v' \in V_{n-1}$, by the edge (v', v_n, α) to obtain a conventional trellis. To avoid such technicalities, the term "conventional trellis" will henceforth refer to a tail-biting trellis with $|V_0| = 1$, thereby considering the class of conventional trellises as a subclass of the class of tail-biting trellises.

Let $\mathbb{F}_q$ be the finite field of q elements, where $q = p^m$ for some prime p and some $m \in \mathbb{N}$, and $\mathbb{N}$ is the set of positive integers. An $[n, k]$ code over $\mathbb{F}_q$ is a k-dimensional subspace of $\mathbb{F}_q^n$. The coordinates of codewords are labeled by indexes from $\{0, 1, \ldots, n-1\}$ throughout this correspondence (e.g., $(c_0, c_1, \ldots, c_{n-1})$). An $[n, k, d]$ code is an $[n, k]$ code of minimum Hamming distance d. Let C be an $[n, k]$ code over $\mathbb{F}_q$, and let $T = (V, E, \mathbb{F}_q)$ be a length-n trellis (either conventional or tail-biting). If $C = C(T)$, we say that T *represents* C.

Let $T = (V, E, \mathbb{F}_q)$ and $T' = (V', E', \mathbb{F}_q)$ be two length-n tail-biting trellises, and let $C := C(T)$, $C' := C(T')$ be the codes represented by T and T'. The *product trellis* [6], [9] $T \times T' = (V^\pi, E^\pi, \mathbb{F}_q)$ is defined in the following way. For each i, $i \in \{0, 1, \ldots, n-1\}$, $V_i^\pi := V_i \times V_i'$. The edge $((v_{i-1}, v'_{i-1}), (v_i, v'_i), \alpha + \alpha')$ is in E^π if

$$(v_{i-1}, v_i, \alpha) \in E \quad \text{and} \quad (v'_{i-1}, v'_i, \alpha') \in E'$$

for $i \in \{1, 2, \ldots, n-1\}$ and $\alpha, \alpha' \in \mathbb{F}_q$. It is straightforward to verify that $C(T \times T') = C(T) + C(T')$.

Let C be an $[n, k]$ code. A set of the form $J := \{\bar{i}, \overline{i+1}, \ldots, \overline{i+l}\}$ (for a positive integer $l < n$ and for some integer i) is called a *span* for a nonzero codeword $\boldsymbol{c} = (c_0, c_1, \ldots, c_{n-1}) \in C$ if $c_j = 0$ for all $j \in \{0, 1, \ldots, n-1\} \backslash J$ (where $A \backslash B$ stands for the complementary set of B in A), and if both $c_{\bar{i}}$ and $c_{\overline{i+l}}$ are nonzero. To simplify notation, we write $[\bar{i}, \overline{i+l}]$ for $\{\bar{i}, \overline{i+1}, \ldots, \overline{i+l}\}$ when $l > 0$, and $[\bar{i}, \bar{i}]$ for $\{\bar{i}\}$. A span $[i_1, i_2]$ $(i_1, i_2 \in \{0, 1, \ldots, n-1\})$ for a nonzero codeword $\boldsymbol{c} \in C$ is called a *conventional span* whenever $i_1 \leq i_2$, and a *cyclic span* otherwise. Note that given a nonzero codeword $\boldsymbol{c} \in C$, there is a unique conventional span for $\boldsymbol{c}$, but usually multiple cyclic spans for $\boldsymbol{c}$. Let $J := [i_1, i_2]$ be a span for a nonzero codeword $\boldsymbol{c} \in C$. We call $\mathrm{Start}\,(J) := i_1$ the *start index* of J, and $\mathrm{End}\,(J) := i_2$ the *end index* of J. Also, the *active interval* of J is defined as $\mathrm{Active}\,(J) := [\overline{i_1+1}, i_2]$ if $i_1 \neq i_2$. If $i_1 = i_2$, then $\mathrm{Active}\,(J)$ is defined to be the empty set. Once a span $J = [i_1, i_2]$ for a nonzero codeword $\boldsymbol{c}$ is chosen, it is possible to assign an *elementary trellis* [6], [9] over $\mathbb{F}_q$, $T_{\boldsymbol{c}}$, to $\boldsymbol{c}$, having the following two properties. First, the subspace $\langle \boldsymbol{c} \rangle \subset \mathbb{F}_q^n$ generated by $\boldsymbol{c}$ is represented by $T_{\boldsymbol{c}}$, i.e., $\langle \boldsymbol{c} \rangle = C(T_{\boldsymbol{c}})$. Second, for all $i \in \{0, 1, \ldots, n-1\}$, $s_i(T_{\boldsymbol{c}}) = 1$ if $i \in \mathrm{Active}\,(J)$, and $s_i(T_{\boldsymbol{c}}) = 0$ otherwise. In particular, $T_{\boldsymbol{c}}$ is a conventional trellis if and only if J is a conventional span.

A linear tail-biting trellis over $\mathbb{F}_q$ is defined as a tail-biting trellis that can be decomposed to a product of elementary trellises over $\mathbb{F}_q$.[1] For integer $\hat{k} \geq k$, let $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{\hat{k}}\}$ be a set that generates C as an $\mathbb{F}_q$-space, and suppose that spans are assigned to the elements of $\mathcal{G}$. Then the (linear) trellis $T := T_{\boldsymbol{g}_1} \times T_{\boldsymbol{g}_2} \times \cdots \times T_{\boldsymbol{g}_{\hat{k}}}$ represents C. Note that if $\hat{k} > k$, then there is a basis for C in $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{\hat{k}}\}$, and we can assume without loss of generality (w.l.o.g.) that $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_k\}$ is such a basis. Clearly, the (linear) tail-biting trellis $T' := T_{\boldsymbol{g}_1} \times T_{\boldsymbol{g}_2} \times \cdots \times T_{\boldsymbol{g}_k}$ represents C, and any lower bound on the state complexity profile of T' is also a lower bound on the state complexity profile of T. Thus, in order to find lower bounds on the state complexity of a linear tail-biting trellis representing C, it is sufficient to consider linear tail-biting trellises corresponding to bases of C.

Let $T(C)$ be the unique conventional minimal trellis of C (see [15] and the references therein). The state complexity of C at index i, $s_i(C)$ $(i \in \{0, 1, \ldots, n\})$, the state complexity profile of C, $\boldsymbol{s}(C)$, and the state complexity $s(C)$ of C are defined as the respective quantities of $T(C)$.

The cut-set bound [2], [17] on the state complexity profile of a (linear or nonlinear) tail-biting trellis T representing an $[n, k]$ code C is $s_0(T) + s_i(T) \geq s_i(C)$ for all $i \in \{1, 2, \ldots, n-1\}$. As a consequence, $s(T) \geq s(C)/2$. In particular, as the state complexity of a linear tail-biting trellises is an integer, we have $s(T) \geq \lceil s(C)/2 \rceil$ for a linear tail-biting trellis T representing C. This bound is referred to as the square-root bound. An additional method to calculate a lower bound on $s(T)$ is to use an algorithm of Kötter and Vardy [6], [8]. This algorithm finds the minimum possible average state complexity of a linear tail-biting trellis representing C in $O(n^2)$ vector operations.

Let C be an $[n, k]$ code. The *support* of a codeword $\boldsymbol{c} \in C$ is defined as $\mathrm{supp}(\boldsymbol{c}) := \{i \in \{0, 1, \ldots, n-1\} | c_i \neq 0\}$. The support of the code C is defined as $\mathrm{supp}(C) := \cup_{\boldsymbol{c} \in C}\{\mathrm{supp}(\boldsymbol{c})\}$. The *support size* of C is $|\mathrm{supp}(C)|$. Let I be a subset of $\{0, 1, \ldots, n-1\}$. The subcode $C_I \subseteq C$ is defined as $C_I := \{\boldsymbol{c} \in C | \mathrm{supp}(\boldsymbol{c}) \subseteq I\}$. Clearly, C_I is a linear subcode of C.

Let $\mathcal{S}_n$ be the group of permutations on $\{0, 1, \ldots, n-1\}$, and define an action of $\mathcal{S}_n$ on $\mathbb{F}_q^n$ by

$$\tau(c_0, c_1, \ldots, c_{n-1}) := (c_{\tau(0)}, c_{\tau(1)}, \ldots, c_{\tau(n-1)})$$

for $\tau \in \mathcal{S}_n$ and $(c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}_q^n$. For an $[n, k]$ code C over $\mathbb{F}_q$ and for $\tau \in \mathcal{S}_n$, set $\tau C := \{\tau \boldsymbol{c} | \boldsymbol{c} \in C\}$. Two $[n, k]$ codes C and D for which there is some $\tau \in \mathcal{S}_n$ such that $D = \tau C$ are called *equivalent*. The state complexity profiles (and, in particular, the state complexities) of equivalent codes may be significantly different from each other (see [15] and the references therein). In the context of conventional trellises, the *minimum state complexity* of an $[n, k]$ code C, $s_{\min}(C)$, is defined as $\min\{s(\tau C) | \tau \in \mathcal{S}_n\}$. In other words, the minimum state complexity is the smallest conventional state complexity that can be achieved by changing the coordinate ordering of C. Let $(k_0(C), k_1(C), \ldots, k_n(C))$ be the *dimension/length profile (DLP)* of C [3], i.e, $k_i(C)$ is the maximum dimension of a subcode of C whose support size is not larger than i, $i \in \{0, 1, \ldots, n\}$. Then the *DLP bound* on the state complexity profile of C is [3]

$$s_i(\tau C) \geq s_i^{\mathrm{DLP}}(C) := k - k_i(C) - k_{n-i}(C)$$

for every $i \in \{0, 1, \ldots, n-1\}$ and every $\tau \in \mathcal{S}_n$. The DLP bound on the minimum state complexity of C is

$$s_{\min}(C) \geq s^{\mathrm{DLP}}(C) := \max\{s_i^{\mathrm{DLP}}(C) | i \in \{0, 1, \ldots, n-1\}\}.$$

[1]Actually, the original definition of a linear tail-biting trellis [6] is different, but the equivalence of the two definitions was proved by Kötter and Vardy [7].

The *generalized Hamming weight hierarchy* (*GHW hierarchy*) [16] of C will be denoted by $\{d_1(C), d_2(C), \ldots, d_k(C)\}$, where $d_j(C)$ is the minimum support size of a j-dimensional subcode of C, $j \in \{1, 2, \ldots, k\}$.

In the context of linear tail-biting trellises, there are currently three different bounds on the state complexity under arbitrary coordinate orderings. The first one, $s(T) \geq s^{\mathrm{DLP}}(C)/2$ if T is a (linear or nonlinear) tail-biting trellis representing C, is a straightforward application of the square-root bound. The second bound, referred to as the total-span bound [2], is $s(T) \geq \lceil k(d-1)/n \rceil$, where T is a linear tail-biting trellis representing C. The total-span bound is easy to calculate (as it requires only the parameters $[n, k, d]$ of the code) but appears to be loose [13]. The third bound, which is an immediate consequence of a theorem of Bocharova *et al.* [1], will be discussed in Sections III and IV.

III. The Generalized Total-Span Bound

In this section, a generalization of the total-span bound is derived. It is shown by an example that the new bound may be tighter than any of the existing bounds.

Theorem 1 (Generalized Total-Span Bound for any Coordinate Ordering): Let C be an $[n, k]$ code, and let T be a linear tail-biting trellis representing C. Then for any $r \in \{1, 2, \ldots, k\}$, the average state complexity of T is lower-bounded by

$$\frac{1}{n}\sum_{i=0}^{n-1} s_i(T) \geq \frac{k(d_r(C)-1)}{n} - (r-1). \tag{1}$$

In particular

$$s(T) \geq \left\lceil \frac{k(d_r(C)-1)}{n} \right\rceil - (r-1). \tag{2}$$

Proof: As mentioned in Section II, it is sufficient to consider the case of $T = T_{\boldsymbol{g}_1} \times T_{\boldsymbol{g}_2} \times \cdots \times T_{\boldsymbol{g}_k}$, where $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_k\}$ is a basis for C, and $T_{\boldsymbol{g}_j}$ is the elementary trellis associated with $\boldsymbol{g}_j$ and the corresponding span J_j, $j \in \{1, 2, \ldots, k\}$. Since for $r = 1$ the theorem reduces to the total-span bound, it is sufficient to consider the cases where $r \in \{2, 3, \ldots, k\}$.

In order to establish the proof, we shall show that

$$\sum_{j=1}^{k} |\mathrm{Active}(J_j)| \geq k(d_r(C)-1) - n(r-1).$$

Set $\mathfrak{s}_j := \mathrm{Start}(J_j)$ for $j \in \{1, 2, \ldots, k\}$ (thereby defining $\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_k$), and assume w.l.o.g. that $\mathfrak{s}_1 \leq \mathfrak{s}_2 \leq \cdots \leq \mathfrak{s}_k$. In addition, for $j \in \{1, 2, \ldots, k\}$, set

$$\mathfrak{e}_j := \begin{cases} \mathrm{End}\,(J_j), & J_j \text{ is conventional} \\ \mathrm{End}\,(J_j) + n, & J_j \text{ is cyclic} \end{cases}$$

(thereby defining $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_k$).

Let $l_1, l_2, \ldots, l_{r-1}$ be $r-1$ distinct indexes in $\{1, 2, \ldots, k\}$ such that $\mathfrak{e}_{l_1}, \mathfrak{e}_{l_2}, \ldots, \mathfrak{e}_{l_{r-1}}$ are the $r-1$ smallest elements in the sequence $(\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_k)$. For $j \in \{k+1, k+2, \ldots, k+r-1\}$, set $\mathfrak{e}_j := \mathfrak{e}_{l_{j-k}} + n$ and $\mathfrak{s}_j := \mathfrak{s}_{l_{j-k}} + n$ (thereby defining $\mathfrak{e}_{k+1}, \mathfrak{e}_{k+2}, \ldots, \mathfrak{e}_{k+r-1}$ and $\mathfrak{s}_{k+1}, \mathfrak{s}_{k+2}, \ldots, \mathfrak{s}_{k+r-1}$). In addition, for $j \in \{k+1, k+2, \ldots, k+r-1\}$, we define $J_j := J_{l_{j-k}}$ and $\boldsymbol{g}_j := \boldsymbol{g}_{l_{j-k}}$.

We now define reordered sequences $\{\tilde{\mathfrak{s}}_j\}_{j=1}^{k+r-1}$ and $\{\tilde{\mathfrak{e}}_j\}_{j=1}^{k+r-1}$, and respective reordered sequences of spans $\{\tilde{J}_j\}_{j=1}^{k+r-1}$ and generators $\{\tilde{\boldsymbol{g}}_j\}_{j=1}^{k+r-1}$ as follows. Let μ be a permutation on $\{1, 2, \ldots, k+r-1\}$ such that $\mathfrak{e}_{\mu(j)} \geq \mathfrak{e}_{\mu(j-1)}$ for all $j \in \{2, 3, \ldots, k+r-1\}$. Set $\tilde{\mathfrak{e}}_j := \mathfrak{e}_{\mu(j)}$, $\tilde{\mathfrak{s}}_j := \mathfrak{s}_{\mu(j)}$, $\tilde{J}_j := J_{\mu(j)}$, and $\tilde{\boldsymbol{g}}_j := \boldsymbol{g}_{\mu(j)}$ for all $j \in \{1, 2, \ldots, k+r-1\}$. Since

$$\sum_{j=1}^{k} \tilde{\mathfrak{e}}_{j+r-1} = \sum_{j=1}^{k} \mathfrak{e}_j + (r-1)n$$

it follows that

$$\begin{aligned}\sum_{i=0}^{n-1} s_i(T) &= \sum_{j=1}^{k} |\mathrm{Active}\,(J_j)| = \sum_{j=1}^{k} \mathfrak{e}_j - \sum_{j=1}^{k} \mathfrak{s}_j \\ &= \sum_{j=1}^{k} (\tilde{\mathfrak{e}}_{j+r-1} - \mathfrak{s}_j) - (r-1)n. \end{aligned} \tag{3}$$

From the definition of $\mathfrak{s}_j$, $j \in \{1, 2, \ldots, k\}$, there are at most $j-1$ indexes $m \in \{1, 2, \ldots, j+r-1\}$ such that $\tilde{\mathfrak{s}}_m < \mathfrak{s}_j$. In other words, there are at least $j+r-1-(j-1) = r$ indexes $m \in \{1, 2, \ldots, j+r-1\}$ such that $\tilde{\mathfrak{s}}_m \geq \mathfrak{s}_j$. Let $m_1 < m_2 < \cdots < m_r \leq j+r-1$ be such r indexes. If $\tilde{\mathfrak{e}}_{j+r-1} - \mathfrak{s}_j < n$ then, since

$$\mathfrak{s}_j \leq \min\{\tilde{\mathfrak{s}}_{m_1}, \tilde{\mathfrak{s}}_{m_2}, \ldots, \tilde{\mathfrak{s}}_{m_r}\}$$

and

$$\tilde{\mathfrak{e}}_{m_r} = \max\{\tilde{\mathfrak{e}}_{m_1}, \tilde{\mathfrak{e}}_{m_2}, \ldots, \tilde{\mathfrak{e}}_{m_r}\} \leq \tilde{\mathfrak{e}}_{j+r-1}$$

it follows that

$$\tilde{\mathfrak{e}}_{m_r} - \min\{\tilde{\mathfrak{s}}_{m_1}, \tilde{\mathfrak{s}}_{m_2}, \ldots, \tilde{\mathfrak{s}}_{m_r}\} < n.$$

Consequently, $\tilde{\boldsymbol{g}}_{m_1}, \tilde{\boldsymbol{g}}_{m_2}, \ldots, \tilde{\boldsymbol{g}}_{m_r}$ are r distinct generators, and span a subcode $D \subset C$ with

$$\begin{aligned} d_r(C) &\leq |\mathrm{supp}(D)| \\ &\leq \overline{\mathrm{End}(\tilde{J}_{m_r}) - \min\{\tilde{\mathfrak{s}}_{m_1}, \tilde{\mathfrak{s}}_{m_2}, \ldots, \tilde{\mathfrak{s}}_{m_r}\}} + 1 \\ &\leq \tilde{\mathfrak{e}}_{j+r-1} - \mathfrak{s}_j + 1. \end{aligned}$$

Hence, if $\tilde{\mathfrak{e}}_{j+r-1} - \mathfrak{s}_j < n$, then $\tilde{\mathfrak{e}}_{j+r-1} - \mathfrak{s}_j \geq d_r(C) - 1$. If $\tilde{\mathfrak{e}}_{j+r-1} - \mathfrak{s}_j \geq n$, then clearly $\tilde{\mathfrak{e}}_{j+r-1} - \mathfrak{s}_j > d_r(C) - 1$, because $n > d_r(C) - 1$ for any $r \in \{1, 2, \ldots, k\}$. We conclude that

$$\tilde{\mathfrak{e}}_{j+r-1} - \mathfrak{s}_j \geq d_r(C) - 1, \qquad \text{for all } j \in \{1, 2, \ldots, k\}$$

and the proof follows from (3). □

Remark 2: A slightly modified version of the bound of Bocharova, Johannesson, Kudryashov, and Ståhl (BJKS) [1, Theorem 4], reads

$$s(T) \geq \left\lceil \frac{k d_r(C)}{n} \right\rceil - r \tag{4}$$

for any $r \in \{1, 2, \ldots, k\}$ (note, though, that an equivalent to the bound (1) on the average state complexity does not appear in [1]). The bound (4) will be referred to as the *BJKS bound*. Clearly, the BJKS bound is not tighter than the generalized total-span bound, since $d_{r+1}(C) - 1 \geq d_r(C)$ for every $r \in \{1, 2, \ldots, k-1\}$ [16], and for $r = k$ the right-hand side (RHS) of (4) is 0. On the other hand, setting $b_r := k(d_r(C)-1)/n - (r-1)$ and $b'_r := kd_r(C)/n - r$ for $r \in \{1, 2, \ldots, k\}$, it follows that $b_r - b_{r-1} = b'_r - b'_{r-1}$ for $r \in \{2, 3, \ldots, k\}$. This means that the maximum of $\{b_r | r \in \{1, 2, \ldots, k\}\}$ and the maximum of $\{b'_r | r \in \{1, 2, \ldots, k\}\}$ are attained for the same value of r. Since $b_r - b'_r < 1$ for every $r \in \{1, 2, \ldots, k\}$, we infer that the generalized total-span bound is tighter than the BJKS bound by at most 1. In particular, asymptotically, the generalized total-span bound gives the same bound as [1, Theorem 5]. Since the generalized total-span bound is valid also for the average state complexity, it follows that the bound of [1, Theorem 5] is valid also for the *relative average state complexity*, which is defined in an obvious way. In any case, it should be emphasized that whereas the BJKS bound is valid for certain cases of *sectionalization* [15], the generalized total-span bound refers to nonsectionalized trellises.

To show that the generalized total-span bound may be tighter than the BJKS bound, we have the following example.

Example 3: Let $m \geq 3$ be an integer, and let $C := \mathrm{RM}\,(1, m)$ be the first-order Reed–Muller code. Then C is a $[2^m, m+1]$ code, and it is known [16] that $d_r(C) = 2^m - 2^{m-r}$ for $r \in \{1, 2, \ldots, m\}$. Let T be a linear tail-biting trellis representing C. Set $b_r := (m+1)(d_r(C) - 1)/2^m - (r-1)$. It follows from Theorem 1 that $s(T) \geq \lceil b_r \rceil$ for every $r \in \{1, 2, \ldots, m\}$. To find the index r for which b_r is maximal, let us check the difference $\delta_r := b_r - b_{r-1}$, for $r \in \{2, 3, \ldots, m\}$. Since $\delta_r = (m+1)(d_r(C) - d_{r-1}(C))/2^m - 1$, it follows that $\delta_r \geq 0$ iff $m + 1 \geq 2^r$, that is, iff $r \leq \lfloor \log_2(m+1) \rfloor$. So the maximum value of b_r is achieved for $r = r^\star := \lfloor \log_2(m+1) \rfloor$, and is equal to

$$\begin{aligned} b_{r^\star} &= (m+1)(1 - 2^{-\lfloor \log_2(m+1) \rfloor} - 2^{-m}) - \lfloor \log_2(m+1) \rfloor + 1 \\ &= m + 2 - (m+1)2^{-\lfloor \log_2(m+1) \rfloor} \\ &\quad - (m+1)2^{-m} - \lfloor \log_2(m+1) \rfloor. \end{aligned}$$

Consequently, $s(T) \geq \lceil b_{r^\star} \rceil = m + 1 - \lfloor \log_2(m+1) \rfloor$.

A similar calculation for the BJKS bound gives $s(T) \geq m - \lfloor \log_2(m+1) \rfloor$. In other words, for the first-order Reed–Muller code, the generalized total-span bound of Theorem 1 surpasses the BJKS bound by 1.

In [11], it was shown that when $C = \mathrm{RM}\,(1, m)$ is arranged in the *standard bit order* [5], the state complexity of any linear tail-biting trellis representing C is at least $m - 1$. If the generalized total-span bound is tight, then there exists a permutation $\tau \in \mathcal{S}_{2^m}$ and a linear tail-biting T representing τC with $s(T)$ of the order of $m - \log_2(m)$. It would be interesting to either find such a permutation τ or, on the contrary, to tighten the generalized total-span bound.

Whereas the original total-span bound may be looser than the square-root bound for an arbitrary coordinate ordering (cf. [13]), the generalized total-span bound is never smaller than the square-root bound. Specifically, we have the following proposition.

Proposition 4: Let C be an $[n, k]$ code. Then

$$\max \left\{ \frac{k(d_r(C) - 1)}{n} - (r-1) | r \in \{1, 2, \ldots, k\} \right\} \geq s^{\mathrm{DLP}}(C)/2.$$

Proof: Let u be a positive integer, $u \leq n - 1$. Put $b_u := ku/n - k_u(C)$. Then

$$b_u + b_{n-u} = k - k_u(C) - k_{n-u}(C) = s_u^{\mathrm{DLP}}(C).$$

Therefore, if the maximum of the DLP bound on the state complexity profile is achieved at index $u_0 \in \{1, 2, \ldots, n-1\}$ (i.e., $s^{\mathrm{DLP}}(C) = s_{u_0}^{\mathrm{DLP}}(C)$), then $\max\{b_{u_0}, b_{n-u_0}\} \geq s^{\mathrm{DLP}}(C)/2$. In particular, the maximum of the sequence $(b_1, b_2, \ldots, b_{n-1})$ is not smaller than $s^{\mathrm{DLP}}(C)/2$. To complete the proof, note that the maximum of $(b_1, b_2, \ldots, b_{n-1})$ is necessarily achieved at an index $u = d_r(C) - 1$ for some $r \in \{1, 2, \ldots, k-1\}$, for which

$$b_u = \frac{k(d_r(C) - 1)}{n} - (r-1). \qquad \square$$

IV. The Generalized Cut-Set Bound

In this section, a generalization of the cut-set bound for linear trellises is derived. It is shown that this bound may be tighter than any of the existing bounds. The following theorem can be considered as an extension of the ideas from [1, Theorem 4].

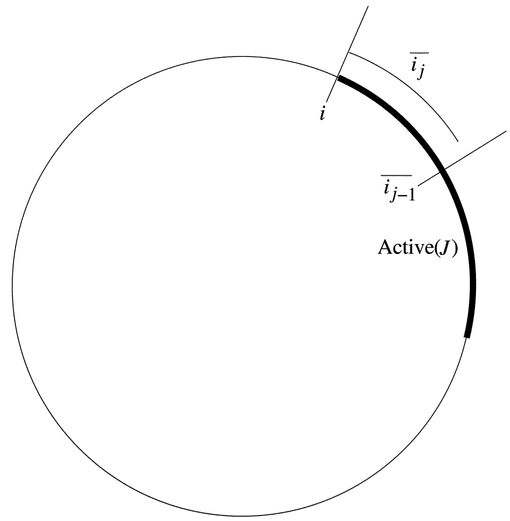

Fig. 1. The case where $\overline{i_{j-1}} \in \mathrm{Active}\,(J)$. The bold line represents $\mathrm{Active}\,(J)$ on the circular time axis. Since $\overline{i_j} \in [i, \overline{i_{j-1} - 1}]$, $\overline{i_j}$ is essentially in $\mathrm{Active}\,(J)$.

Theorem 5 (Generalized Cut-Set Bound for a Fixed Coordinate Ordering): Let C be an $[n, k]$ code, and let T be a linear tail-biting trellis representing C. Then for $l + 1$ integers $i_0 = 0, i_1, i_2, \ldots, i_l \geq n$ (where $l \in \mathbb{N}$) satisfying

$$i_{j-1} < i_j < i_{j-1} + n$$

for all $j \in \{1, 2, \ldots, l\}$, it holds that

$$\sum_{j=1}^{l} s_{\overline{i_j}}(T) + \sum_{j=0}^{l-1} \dim C_{I_j} \geq k \left\lfloor \frac{i_l}{n} \right\rfloor \tag{5}$$

where $I_j := [\overline{i_j}, \overline{i_{j+1} - 1}]$ for $j \in \{0, 1, \ldots, l-1\}$.

Proof: As before, it is sufficient to consider the case of $T = T_{\boldsymbol{g}_1} \times T_{\boldsymbol{g}_2} \times \cdots \times T_{\boldsymbol{g}_k}$, where $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_k\}$ is a basis for C, and $T_{\boldsymbol{g}_j}$ is the elementary trellis associated with $\boldsymbol{g}_j$ and its span J_j, $j \in \{1, 2, \ldots, k\}$. Let $\mathcal{J} := \{J_1, J_2, \ldots, J_k\}$ be the set of spans used to define T. We define a family $\{\mathcal{J}_m\}_{m=0}^{2l}$ of subsets of $\mathcal{J}$ as follows:

$$\mathcal{J}_m := \begin{cases} \{J \in \mathcal{J} | \overline{i_{m/2}} \in \mathrm{Active}\,(J)\}, & m \text{ is even} \\ \{J \in \mathcal{J} | J \subseteq I_{\lfloor m/2 \rfloor}\}, & m \text{ id odd} \end{cases}$$

for every $m \in \{0, 1, \ldots, 2l\}$. Now we claim that

$$\sum_{m=1}^{2l} |\mathcal{J}_m| \geq k \left\lfloor \frac{i_l}{n} \right\rfloor. \tag{6}$$

To prove this claim, it is sufficient to show that every span $J \in \mathcal{J}$ appears in at least $\lfloor (i_l - i)/n \rfloor + \lceil i/n \rceil (\geq \lfloor i_l/n \rfloor)$ sets in $\{\mathcal{J}_m\}_{m=1}^{2l}$, where $i \in \{0, 1, \ldots, n-1\}$ is such that $\overline{i-1}$ is the start index of J. Let $j \in \{1, 2, \ldots, l\}$ be such that $\lfloor (i_j - i)/n \rfloor > \lfloor (i_{j-1} - i)/n \rfloor$. Then $\overline{i_j} \in [i, \overline{i_{j-1} - 1}]$ (since $0 < i_j - i_{j-1} < n$). If $\overline{i_{j-1}} \in \mathrm{Active}\,(J)$, then essentially $\overline{i_j} \in \mathrm{Active}\,(J)$, because i is the start index of $\mathrm{Active}\,(J)$ (see Fig. 1). So in this case, $J \in \mathcal{J}_{2j}$. If $\overline{i_{j-1}} \notin \mathrm{Active}\,(J)$, then either $\overline{i_j} \in \mathrm{Active}\,(J)$, in which case $J \in \mathcal{J}_{2j}$, or $\overline{i_j} \notin \mathrm{Active}\,(J)$, in which case $J \subseteq I_{j-1}$ and, therefore, $J \in \mathcal{J}_{2j-1}$ (see Fig. 2). So, for any $j \in \{1, 2, \ldots, l\}$ such that

$$\lfloor (i_j - i)/n \rfloor > \lfloor (i_{j-1} - i)/n \rfloor$$

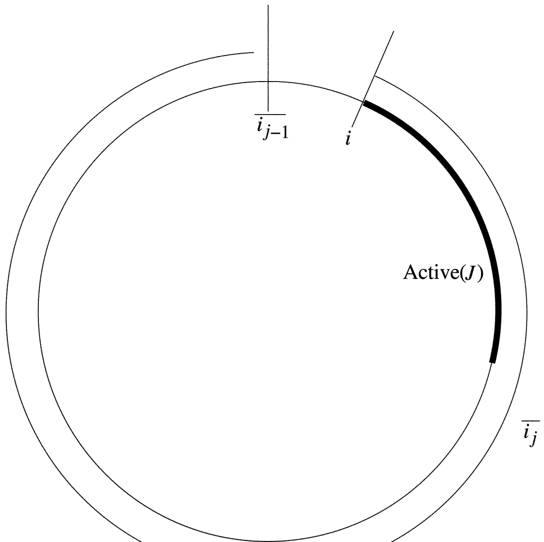

Fig. 2. The case where $\overline{i_{j-1}}$ is not in $\mathrm{Active}(J)$. As before, $\overline{i_j} \in [i, \overline{i_{j-1}} - 1]$, and, therefore, $\overline{i_j}$ can be either in $\mathrm{Active}(J)$, or outside of $\mathrm{Active}(J)$, in which case $J \subseteq I_{j-1} = [\overline{i_{j-1}}, \overline{i_j} - 1]$.

either $J \in \mathcal{J}_{2j}$ or $J \in \mathcal{J}_{2j-1}$. Since there is a total of $\lfloor (i_l - i)/n \rfloor + \lceil i/n \rceil$ such indexes j, the claim is established.

Now, for even m we have

$$|\mathcal{J}_m| = |\{J \in \mathcal{J} | \overline{i_{m/2}} \in \mathrm{Active}\,(J)\}| = s_{\overline{i_{m/2}}}(T).$$

In addition, for odd m we have

$$|\mathcal{J}_m| = |\{J \in \mathcal{J} | J \subseteq I_{\lfloor m/2 \rfloor}\}| \leq \dim C_{I_{\lfloor m/2 \rfloor}}.$$

The proof therefore follows from (6). □

Remark 6: Let i be an index in $\{1, 2, \ldots, n-1\}$. For a choice of $l := 2$, $i_1 := i$, and $i_2 := n$ in Theorem 5, the theorem reduces to the cut-set bound $s_0(T) + s_i(T) \geq s_i(C)$. Actually, it is possible to prove the following more general claim.

Claim: Let C be an $[n, k]$ code, and let T be a linear tail-biting trellis representing C. Then for integers $0 = \tilde{i}_0 < \tilde{i}_1 < \tilde{i}_2 < \cdots < \tilde{i}_{\tilde{l}} < n$ (where $\tilde{l} \in \mathbb{N}$) it holds that

$$\sum_{j=0}^{\tilde{l}} s_{\tilde{i}_j}(T) \geq \sum_{j=1}^{\tilde{l}} s_{\tilde{i}_j}(C) - \sum_{j=2}^{\tilde{l}} z_C([\tilde{i}_{j-1}, \tilde{i}_j - 1])$$

where for a subset $I \subset \{0, 1, \ldots, n-1\}$, $z_C(I)$ is the dimension of the subcode of C consisting of all codewords $(c_0, c_1, \ldots, c_{n-1}) \in C$ with $c_i = 0$ for all $i \in I$ (i.e., $z_C(I) := \dim C_{\{0,1,\ldots,n-1\} \setminus I}$). In particular

$$s(T) \geq \left\lceil \frac{1}{\tilde{l}+1} \left(\sum_{j=1}^{\tilde{l}} s_{\tilde{i}_j}(C) - \sum_{j=2}^{\tilde{l}} z_C([\tilde{i}_{j-1}, \tilde{i}_j - 1]) \right) \right\rceil. \quad (7)$$

Proof: Set $l := \tilde{l} + 1$, and define $i_1 := \tilde{i}_{\tilde{l}}$, $i_{j+1} := i_j + n - (\tilde{i}_{\tilde{l}+1-j} - \tilde{i}_{\tilde{l}-j})$ for $j \in \{1, 2, \ldots, l-2\}$, and $i_l := n \lceil i_{l-1}/n \rceil$. Since $z_C([\tilde{i}_{j-1}, \tilde{i}_j - 1]) = \dim C_{I_{l-j}}$ (using the terminology of Theorem 5) and $s_i(C) = k - \dim C_{[i,n-1]} - \dim C_{[0,i-1]}$ for any $i \in \{1, 2, \ldots, n-1\}$, the assertion follows from Theorem 5. □

$$G := \begin{pmatrix}
11000111010100000000000000000000000000000000 \\
01100011101010000000000000000000000000000000 \\
00110001110101000000000000000000000000000000 \\
00011000111010100000000000000000000000000000 \\
00001100011101010000000000000000000000000000 \\
00000110001110101000000000000000000000000000 \\
00000011000111010100000000000000000000000000 \\
00000001100011101010000000000000000000000000 \\
00000000110001110101000000000000000000000000 \\
00000000011000111010111000111010100000000000 \\
00000000001100011101011100011101010000000000 \\
00000000000110001110101110001110101000000000 \\
00000000000000000000000011000111010100000000 \\
00000000000000000000000001100011101010000000 \\
00000000000000000000000000110001110101000000 \\
00000000000000000000000000011000111010100000 \\
00000000000000000000000000001100011101010000 \\
00000000000000000000000000000110001110101000 \\
00000000000000000000000000000011000111010100 \\
00000000000000000000000000000001100011101010 \\
00000000000000000000000000000000110001110101
\end{pmatrix}.$$

Fig. 3. The generator matrix for Example 8.

Example 7: Choosing $\tilde{l} = 2$ in (7), we obtain

$$s(T) \geq \left\lceil \frac{1}{3} \left(2s(C) - z_C([\tilde{i}_1, \tilde{i}_2 - 1]) \right) \right\rceil \quad (8)$$

where $\tilde{i}_1$ and $\tilde{i}_2$ may be taken as the first and last index i (respectively) for which $s_i(C) = s(C)$. Checking when the bound of (8) is not tighter than the square-root bound $s(T) \geq \lceil s(C)/2 \rceil$ gives the necessary condition for achieving the square-root bound that was presented in [13].

Example 8: One possible way to find a lower bound on the state complexity of a linear tail-biting trellis for a code C under a fixed coordinate ordering, is to find the minimum possible average state complexity of any linear tail-biting trellis representing C using an algorithm of Kötter and Vardy [6], [8]. In this example, it is shown that there are codes for which the bound of Theorem 5 is larger even than the average state complexity of the conventional trellis. Let C be the $[44, 21]$ code with the generator matrix G defined in Fig. 3. Choosing $i_1 = 33$, $i_2 = 44 + 11$, and $i_3 = 2 \cdot 44$, we have $I_0 = [0, 32]$, $I_1 = [33, 10]$, and $I_2 = [11, 43]$ (using the terminology of Theorem 5). Since $\dim C_{I_0} = \dim C_{I_2} = 10$ and $\dim C_{I_1} = 0$, the generalized cut-set bound reads $\lceil 22/3 \rceil = 8$. The average of the state complexity profile of C is $267/44 < 7$, which means that the bound obtained from the Kötter–Vardy algorithm is certainly not larger than 7. It should be further noted that similar examples can be given for an arbitrary large code length (even a length for which the running time of the Kötter–Vardy algorithm is too long in practice).

Example 9: Let C be an $[n, k]$ cyclic code (under a cyclic coordinate ordering). Put $g := \gcd(k, n) = \gcd(n-k, n)$, and set $l := n/g$. For $j \in \{0, 1, \ldots, l\}$, let $i_j := j(n-k)$. Since $|I_j| = n-k < n-k+1$ (using the terminology of Theorem 5), we have $\dim C_{I_j} = 0$ for $j \in \{0, 1, \ldots, l-1\}$. It therefore follows from Theorem 5 that if T is a linear tail-biting trellis representing C, then

$$s(T) \geq \left\lceil \frac{k(n-k)}{n} \right\rceil$$

which is exactly the bound of [11]. Note that there exist cases for which this bound is tight [12].

Theorem 5 immediately implies the following bound for any coordinate ordering.

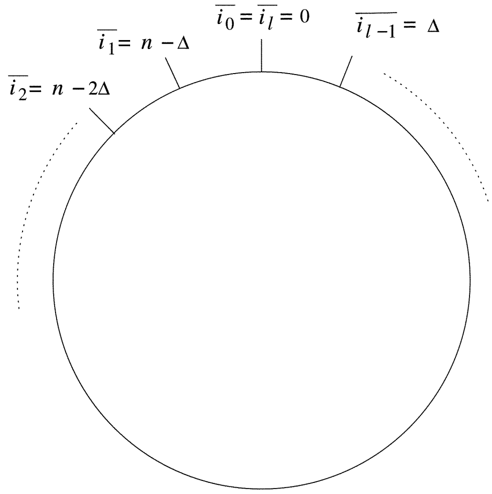

Fig. 4. The indices in Example 11.

Theorem 10 (Generalized Cut-Set Bound for any Coordinate Ordering): Notation as in Theorem 5.

$$\sum_{j=1}^{l} s_{\overline{i_j}}(T) + \sum_{j=0}^{l-1} k_{|I_j|}(C) \geq k \left\lfloor \frac{i_l}{n} \right\rfloor. \qquad (9)$$

Clearly, the state complexity of a sectionalized linear tail-biting trellis representing a linear code C can be lower than the generalized total-span bound or even the BJKS bound (i.e., both of these bounds are not valid for arbitrary sectionalization). For example, the state complexity of a conventional trellis with two equi-length sections for the *direct sum* [10, p. 76] of C with itself can be as low as 0. One of the applications of Theorem 10 is the derivation of bounds on the state complexity of sectionalized tail-biting trellises with arbitrary section lengths (either equal or unequal). In the following example, we focus on sectionalized trellises with equi-length sections.

Example 11: Let C be an $[n, k]$ code, let Δ be a positive integer dividing n, and let T be a linear tail-biting trellis representing C. Set $l := n/\Delta$ and $i_j := j(n-\Delta)$ for $j \in \{0, 1, \ldots, l\}$, as illustrated in Fig. 4. Using the terminology of Theorem 10, we have $|I_j| = n - \Delta$ for all $j \in \{0, 1, \ldots, l-1\}$. It therefore follows from Theorem 10 that

$$\max\{s_{j\Delta}(T) | j \in \{1, 2, \ldots, l\}\} \geq \left\lceil \frac{k(n-\Delta)}{n} - k_{n-\Delta}(C) \right\rceil.$$

Let us now consider the case $C = \text{RM}\,(1, m)$. Choosing

$$\Delta = n - d_{r^\star}(C) = 2^{m - r^\star}$$

where $r^\star := \lfloor \log_2(m+1) \rfloor$ (as in Example 3), we obtain

$$\max\{s_{j\Delta}(T) | j \in \{1, 2, \ldots, l\}\} \geq m - \lfloor \log_2(m+1) \rfloor. \qquad (10)$$

Note that the RHS of (10) is equal to the BJKS bound (cf. Example 3). This means that the state complexity of a sectionalized tail-biting trellis with equi-length sections representing RM $(1, m)$ can be smaller than this bound only if the section length is larger than $2^{m - \lfloor \log_2(m+1) \rfloor}$ (that is, the number of sections is smaller than $2^{\lfloor \log_2(m+1) \rfloor} \leq m + 1$).

We conclude by noting that an alternative, less direct, proof of the generalized total-span bound (Theorem 1) can be deduced from Theorem 5. Using the terminology of Theorem 5, this can be done by setting $|I_j| = d_r(C) - 1$ for all $j \in \{0, 1, \ldots, n/g - 1\}$, and then applying the theorem to g cyclic shifts of T, where $g := \gcd(n, d_r(C) - 1)$.

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