Concatenated Multilevel Block Coded Modulation

Hanan Herzberg, Yair Be'ery, *Member, IEEE,* and Jakov Snyders

***Abstract*— Encoding and decoding schemes for concatenated multilevel block codes are presented. By one of these structures, a real coding gain of 5.6–7.4 dB for the bit error range of 10^{-6} – 10^{-9} is achieved for transmission through the additive white Gaussian noise channel. Also, a rather large asymptotic coding gain is obtained. The new coding schemes have very low decoding complexity and increased coding gain in comparison with the conventional block and trellis coded modulation structures. A few design rules for concatenated (single and) multilevel block codes with large coding gain are also provided.**

I. Introduction

CODED modulation has been recognized as an efficient scheme for transmitting information through band limited channels (e.g., [1]–[8]). The main idea of the conventional coded modulation scheme is the combination of error correction coding with modulation, in a way which provides the largest squared Euclidean free distance between the transmitted signal sequences. The fact that large squared Euclidean distance leads to large asymptotic (i.e., for signal to noise ratio approaching infinity) coding gain, was the motivation for Ungerboeck's schemes [1]. Researchers intended at first (e.g., [2]) to find schemes with large asymptotic coding gain, without paying much attention to the coding gain for moderate signal to noise ratio. Forney *et al.* [3] realized that a significant reduction of the coding gain occurs at bit error rate (BER) of 10^{-6}, a practically accepted rate in many communication systems. This observation led to the concept of the "error coefficient" [3], namely the order of the number of coded sequences at minimum distance from an average transmitted sequence. The reduction of the coding gain at moderate signal-to-noise ratio is explained mainly by the error coefficient. The effective coding gain, which has become an important parameter of a coded modulation scheme, is the computed coding gain for BER of 10^{-6} based on the rule of thumb that an increase by a factor of two in the error coefficient reduces the coding gain by 0.2 dB [4] (note that in this paper we are concerned with additive white Gaussian noise channel). It is clear that the effective coding gain does not represent accurately the real coding gain [9]. For the latter, one should perform an exact analysis or a computer simulation at the desired error rate. The real coding gain may be approximated by using the dominant terms of the union bound for the probability of decoding error (see, e.g., [9] [10]). Such bound is not tight enough when the error coefficient is large and the signal-to-noise ratio is small.

Paper approved by the editor for Communications Theory of the IEEE Communications Society. Manuscript received July 25, 1990; revised February 12, 1991 and August 27, 1991. This paper was presented in part at the International Conference on New Trends in Communications, Control, and Signal Processing, Bilkent University, Ankara, Turkey, July 1990.

The authors are with the Department of Electrical Engineering, Tel-Aviv University, Ramat-Aviv 69978, Israel.

IEEE Log Number 9206075.

The idea of multilevel coding, conceived by Imai and Hirakawa in an early paper [11], is a combination of several error correction codes employing a partition of some signal constellation into subsets. A typical code $\mathbb{C}$ of this kind is based on a signal set S_0 and an L level partition chain $S_0/S_1/\cdots/S_L$ [4]. The sequence of bits associated with the partition S_{i-1}/S_i is determined by the use of the code C_i with minimum Hamming distance d_i, $i = 1, 2, \cdots, L$. These L sequences determine the sequence of channel symbols to be transmitted (see, e.g., [4], [11]). The minimum squared Euclidean distance of subset S_i (also named intrasubset distance [7] or subset distance [8]) is denoted δ_i. Obviously $\delta_0 \leq \delta_1 \leq \cdots \leq \delta_L$. The minimum squared Euclidean distance $d(\mathbb{C})$ of $\mathbb{C}$ is given by (see, e.g., [4], [8])

$$d(\mathbb{C}) = \min(d_1\delta_0, d_2\delta_1, \cdots, d_L\delta_{L-1}, \delta_L) \qquad (1)$$

where δ_L is finite in the frequently considered case that some of the bits are uncoded, otherwise $\delta_L = \infty$. Proper choice of the codes and their combination lead to large $d(\mathbb{C})$ and consequently to high asymptotic coding gain, whereas the effective coding gain is strongly affected by the error coefficient.

Another important parameter of the coded modulation scheme is the computational complexity of the decoder. Usually a kind of suboptimal decoder, named multistage decoder, is used for multilevel codes [7], [8], [11], [12]. This decoder employs a separate decoder for each of the codes $C_1, C_2, \cdots, C_L$. The decoding process starts with the decoding of C_1 based on the received signal. The decoding of the code C_i, for each $i = 2, 3, \cdots, L$, is based on the received signal and the estimated codewords of $C_1, C_2, \cdots, C_{i-1}$, without using any information related to the rest of the codes: $C_{i+1}, \cdots, C_L$. This causes the decoder to be suboptimal. However, the cost in the coding gain is moderate (especially for large signal-to-noise ratios), whereas the reduction in complexity is substantial (see [12] for examples).

Using a concatenated code (see, e.g., [14]) instead of each of the binary codes $C_1, C_2, \cdots, C_L$ has been mentioned in [8]. Concatenated coded phase modulation using trellis inner codes is considered in [9]. Phase-invariant concatenated coded modulation consisting of block inner codes and a single outer code is discussed in [10]. For multilevel block coded modulation (BCM) it is convenient to use separate outer codes for each partition level, rather than using one outer code and regarding a multilevel BCM as an inner code. This is partly due to the different correction capability required by the outer code at different partition levels (detailed explanation is deferred to the following sections).

In this paper we are concerned with BCM, in which lattices play a major rule. The use of lattices for this purpose was intensively studied by Forney [5] and Conway and Sloane [13]. Lattice codes based on one of the constructions B, C, and D [13, ch. 5] may be regarded as multilevel codes. Frequently encountered examples are the lattices of the Barnes–Wall family and the Leech lattice [13]. We shall employ a concatenation of such lattice codes and Reed–Solomon (RS) codes.

We shall present a general structure and a few examples of powerful concatenated multilevel codes. A massive reduction in the decoding complexity is achieved by hard decision decoding of the RS outer codes. Furthermore, inclusion of the RS codes in the coded modulation scheme enables the use of relatively simple inner binary codes. However, in spite of the low decoding complexity of the resulting multilevel coding scheme, the coding gain is larger than that of the conventional multilevel code structure. The last statement will be proved for the asymptotic coding gain, whereas for low signal-to-noise ratio it is supported by computer simulation. Computer simulation of the inner binary decoders were very helpful for designing the proper concatenation per each partition level and obtaining high real coding gain. In one of the examples, which is based on 64-QAM constellation, a surprisingly large asymptotic coding gain of 20.8 dB was obtained. The real coding gain of this coding scheme for the BER in the range of $10^{-6} - 10^{-9}$ (an interesting range for many practical applications) is 5.6 – 7.4 dB. Remarkably, the complexity of the decoding for this structure is much less than the complexity of decoding of the Leech lattice [15] and many known trellis codes [4] that have less coding gain for the same range of BER (For details see Section IV).

In the next section we present the basic structure of the encoding and decoding schemes. Three illustrative variations of the basic scheme, Examples 1–3, are also provided. In Section III we examine the performance of the basic scheme, a single-level concatenated scheme and the structures presented by Examples 1–3. In Section III-A the asymptotic coding gain is analyzed, and design rules for achieving high asymptotic coding gain for the general case are provided. Also, the maximum obtainable coding gain, obtained by proper design of the components, is calculated for the Examples 1–3 for various values of effective rate. The real coding gain is addressed in Section III-B and -C. In Section III-B single-level concatenated schemes are studied; whereas Section III-C deals with the multilevel concatenated case. The results, obtained by computer simulation, are helpful for illuminating the constraints to be imposed on the component codes in order to obtain a large real coding gain. Section IV contains discussion, that includes assessment of decoding complexity.

II. Structure of the Concatenated Multilevel Code

The proposed concatenated multilevel code consist of concatenations between binary inner codes and RS outer codes. Fig. 1 shows the structure of the basic encoder, which comprises encoders of L concatenated codes C_i; $i = 1, 2, \cdots, L$. Let N_i, K_i, and D_i be the length, dimension and minimum Hamming distance, respectively, of the outer RS code over

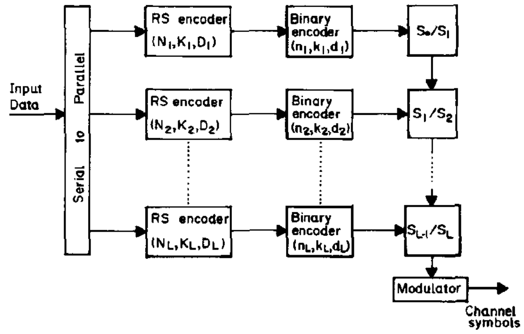


Fig. 1. Basic encoding scheme.

$GF(2^{B_i})$ at partition level i. The parameters of the inner code are n_i, k_i, and d_i. The information bits at the input to the encoder are divided into L sequences of length B_i; $i = 1, 2, \cdots, L$, which form the information symbols for the L RS codes. The codeword of the concatenated code C_i determines the coset of S_i in the partition S_{i-1}/S_i. The rate of the code C_i is given by

$$R_i = \frac{k_i K_i}{n_i N_i}. \tag{2}$$

We wish to point out that Fig. 1 exhibits the basic scheme, which may be modified into more desirable structures, as shown in Examples 1–3. In such modifications a few outer codes are combined with a common inner code.

The channel symbols are taken from a QAM constellation (a subset of a shifted version of lattice Z^2) in a way similar to the one proposed in [7] for two-level partition. For three-level partition, if the signal point (x, y) is labeled by $(a_1 b_1 c_1 a_2 b_2 c_2)$ then

$$\begin{aligned} 2x &\equiv (-1)^{a_1} + 2(-1)^{b_1} + 4(-1)^{c_1} \quad (\text{modulo } 16), \\ 2y &\equiv (-1)^{a_2} + 2(-1)^{b_2} + 4(-1)^{c_2} \quad (\text{modulo } 16). \end{aligned} \tag{3}$$

An N dimensional lattice is determined by $N/2$ such signal points. For the first level of partition, we demand the combination of the bits a_1 and a_2 of these $N/2$ signal points to be a codeword of C_1. In the same fashion b_1, b_2, C_2, and c_1, c_2, C_3 are related to the second and third partition levels, respectively. Each partition is a four way partition and subsets distances are $\delta_0 = 1$, $\delta_1 = 4$, $\delta_2 = 16$, and $\delta_3 = 64$. This method may straightforwardly be extended to partitioning into L levels.

Fig. 2 illustrates the basic structure of the concatenated multistage decoder, which is suited to the basic encoder of Fig. 1. The decoders for the inner codes are soft-decision decoders whereas the outer decoders employ hard decision. The multistage decoder for the conventional multistage coding, which is aimed at achieving massive reduction in the computational complexity, suffers from a moderate reduction in the coding gain (especially for low signal-to-noise ratios). One reason for this reduction is the propagation of decoding errors from stage to stage. Such reduction of the coding gain barely exists in the concatenated multistage decoder. This is due to the correction capability of the outer codes, which provide

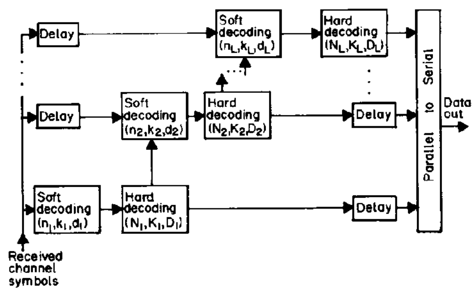


Fig. 2. Basic decoding scheme.

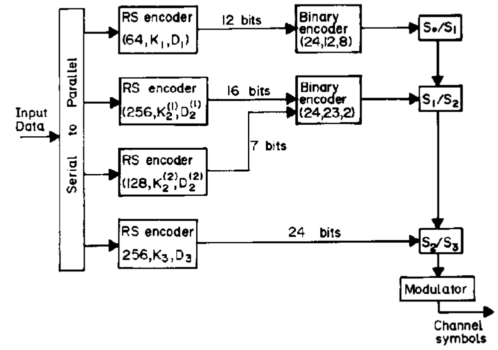


Fig. 3. Encoding scheme for Example 1.

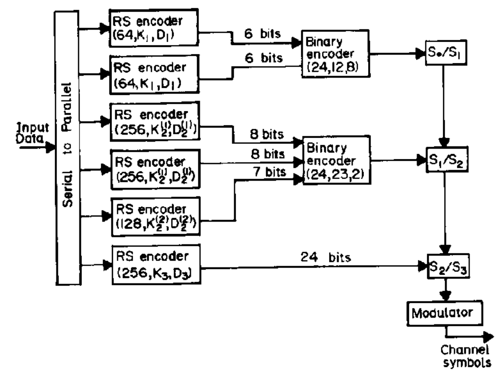


Fig. 4. Modified encoding scheme for Example 1.

reliable information to the soft decoders at all the levels of partitioning.

The following examples illustrate the power and enlighten the structure of the proposed concatenated multilevel coding scheme. These examples are obtained by slight modification of the basic scheme. Only the construction of the codes is presented in this section; the performance is analyzed in the next section.

As inner codes we will use a combination of binary codes that construct some well known lattices [5], [13]. Consequently, the inner codes at all partition levels have the same length n. The lattices we use have large coding gain, which will be increased further by the concatenation with the RS codes, as presented here. For the purpose of matching the length of the RS codes at the various levels, it is convenient to take all of them to be a power of 2. The following examples are based on the 64 QAM constellation, in which each signal point is represented by six bits, labeled by $(a_1b_1c_1a_2b_2c_2)$.

Example 1: Let the inner codes construct the Leech half lattice H_{24} [5], a sublattice of the Leech lattice Λ_{24}. H_{24} is given by

$$H_{24} = C_1(24,12,8) + 2C_2(24,23,2) + 4Z^{24} \qquad (4)$$

where C_1 is the Golay code and C_2 is the single parity check code, both of length 24, and the remaining bits are uncoded. It is convenient to regard the remaining bits as being the output of the universe $(24,24,1)$ code, placed as the binary code at the third partition level. Fig. 3 shows the structure of this concatenated multilevel encoder, which is organized in the following order. 1) At the first level of partition two symbols of RS $(64,K_1,D_1)$ code over $GF(2^6)$ are the source for the information bits of the $(24,12,8)$ Golay codeword. From (2) the rate of C_1 is $R_1 = K_1/128$. 2) At the second partition level the information bits of a binary $(24,23,2)$ parity-check codeword are taken from two symbols of RS $(256,K_2^{(1)},D_2^{(1)})$ over $GF(2^8)$ and one symbol of RS $(128,K_2^{(2)},D_2^{(2)})$ over $GF(2^7)$. The rate of C_2 is given by $R_2 = 23(2K_2^{(1)} + K_2^{(2)})/15,360$. 3) the remaining bits are encoded by the RS $(256,K_3,D_3)$ code over $GF(2^8)$.

The decoder for this structure is similar to that of Fig. 2 but has additional RS decoders (e.g., the decoder for the second-level contains two RS decoders, which match the corresponding encoders). Using more than one symbol of a RS codeword as input to the inner encoder increases the probability of symbol error at the input to the corresponding outer decoder, thereby decreasing the real coding gain. As a countermeasure, we split some of the outer codes (see Fig. 4). Since the uncoded bits are less sensitive to noise (further explanation is deferred to Section III), the RS code at the third level of partition may remain without change.

The construction of the next example is a modification of the previous one, aimed at achieving better performance with less computational complexity.

Example 2: The inner codes of the lattice considered now are obtained by shortening the inner codes of Example 1. At the first level there is the binary $(21,9,8)$ shortened Golay code with the generator matrix obtained by removing the first three rows and columns 8, 16, and 24 of the generator matrix of the Golay code, given in [15]. The inner code for C_2 is the binary $(21,20,2)$ single parity check code and the remaining bits are uncoded. We remark that these codes construct a sublattice of Λ_{24} with kissing number 27,720, which is identical to the kissing number of the sublattice Λ_{21} [13] of Λ_{24}. Fig. 5 presents the encoder for this concatenated multilevel code.

Example 3: In the present scheme the inner codes construct the 16 dimensional Barnes–Wall lattice, Λ_{16}. This lattice may be defined [5] as follows:

$$\Lambda_{16} = C_1(16,5,8) + 2C_2(16,15,2) + 4Z^{16} \qquad (5)$$

where C_1 is a Reed–Muller (RM) code, C_2 is a single parity

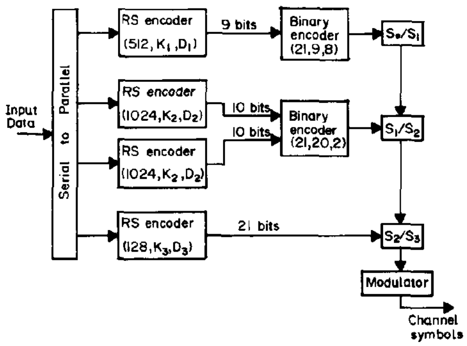


Fig. 5. Encoding scheme for Example 2.

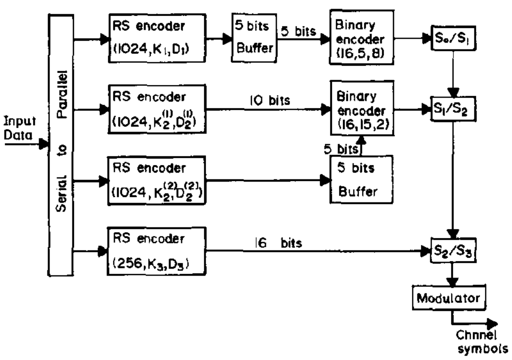


Fig. 6. Encoding scheme for Example 3.

check code and the remaining bits are uncoded. Fig. 6 shows the structure of the encoder. Note that during each time cycle of the outer codes, information bits are provided for two codewords of the inner codes. The task of the buffers is to split each RS symbol of ten bits into two information vectors for the inner codes. The information bits for the inner code at the second partition level consist of 10 and 5 bits from the outer RS $(1024, K_2^{(1)}, D_2^{(1)})$ and RS $(1024, K_2^{(2)}, D_2^{(2)})$ codes, respectively.

As shown in the next section, proper design of the foregoing constructions leads to remarkable performance.

III. Performance Analysis

The decoding fails if the N_i symbols contain more than $t_i = \lfloor (D_i - 1)/2 \rfloor$ errors. For a given probability P_{si} of symbol error at the input of the outer decoder i, the probability P_{bli} of incorrectly decoding the block is given by

$$P_{bli} = \sum_{j=t_i+1}^{N_i} \begin{bmatrix} N_i \\ j \end{bmatrix} P_{si}^j (1 - P_{si})^{N_i - j}. \tag{6}$$

Assuming that most of the block errors are codewords at minimum Hamming distance D_i from the correct codeword, the output BER of the outer decoder at partition level i is closely approximated by

$$P_{bi} \cong \frac{D_i}{2N_i} \sum_{j=t_i+1}^{N_i} \begin{bmatrix} N_i \\ j \end{bmatrix} P_{si}^j (1 - P_{si})^{N_i - j}. \tag{7}$$

The average BER is given by

$$P_b = \frac{\sum_{i=1}^{L} R_i P_{bi}}{\sum_{i=1}^{L} R_i}. \tag{8}$$

Notice that (7) holds for the basic structure, whereas for the modified structures (Examples 1–3) an average BER have to be considered. This average is taken over the error rates of the decoders at each partition level.

A. Asymptotic Coding Gain

The asymptotic coding gain (also called fundamental coding gain [4]) of the basic scheme and the examples of Section II is analyzed for the AWGN channel. As for the conventional multilevel coding, the coding gain is reduced by 3 dB per each redundant bit per channel symbol [3]. The asymptotic coding gain is an upper bound to the real coding gain. Although this bound is not tight, particularly for concatenated codes, it may serve as a rather close approximation for the real coding gain achieved by simple structures (e.g., the single level schemes discussed in Section III-B).

The number of information bits per channel symbol, named effective rate and denoted by R_{eff}, is

$$R_{\text{eff}} = 2 \sum_{i=1}^{L} R_i \tag{9}$$

where R_i is given by (2) and the factor 2 stems from the presence of two bits per each partition in the labeling of a signal point. The number of redundant bits per channel symbol, denoted by $\rho(\mathbb{C})$ (also called normalized redundancy [5]), is

$$\rho(\mathbb{C}) = \log_2 M - R_{\text{eff}} \tag{10}$$

where M is the size of the QAM constellation.

For large signal-to-noise ratio the first term in the right side of (7) is dominant, hence

$$P_{bi} \cong \frac{D_i}{2N_i} \begin{bmatrix} N_i \\ t_i + 1 \end{bmatrix} P_{si}^{t_i+1}. \tag{11}$$

Since the correction capability of the RS codes reduces sharply the error propagation from one partition level to the next, we may use the following approximation [16]

$$P_{si} \approx \alpha_i Q\left(\sqrt{\delta_{i-1} d_i R_{\text{eff}} E_b / 2N_0}\right) \tag{12}$$

where E_b is energy of an information bit, N_0 is the noise power spectral density, $Q(\cdot)$ is the complementary error function defined by $Q(x) \equiv \left(1/\sqrt{2\pi}\right) \int_x^\infty e^{-y^2/2}\,dy$ and α_i is a constant. Recall that, $Q(x)$ is closely approximated by $Q(x) \approx \frac{1}{2} e^{-x^2/2}$ for large values of x. Thus,

$$P_{si} \approx (\alpha_i/2) e^{-\delta_{i-1} d_i R_{\text{eff}} E_b / 4N_0}. \tag{13}$$

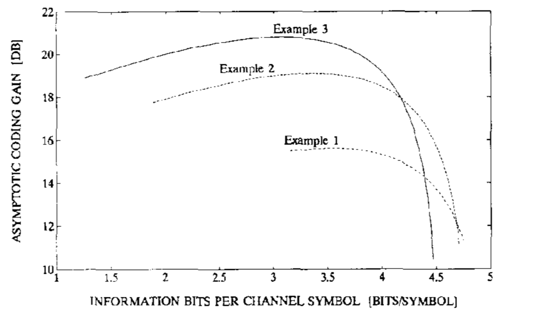


Fig. 7. *Maximum obtainable coding gain for various values of* R_{eff}.

Substitution of (13) into (11) yields

$$P_{bi} \approx \beta_i e^{-\delta_{i-1} d_i (t_i+1) R_{\text{eff}} E_b / 4N_0} \tag{14}$$

where β_i is a constant. We shall compare our system with a $2^{R_{\text{eff}}}$–QAM uncoded scheme operating at the same average BER. The BER of the reference system P_{br} is approximately

$$P_{br} \approx \beta e^{-\delta R_{\text{eff}} E_b / 4N_0} \tag{15}$$

where δ is the minimum squared Euclidean distance between the signal points in the reference system and β is a constant. Assuming normalized constellation, $\delta_0 = 1$ (see Section II), δ is the cost of the redundancy, which according to [3] is given (in dB) by

$$\delta = 3\rho(\mathbb{C}). \tag{16}$$

Due to (8), (14)–(16) the asymptotic coding gain γ is given by

$$\gamma = 10\log_{10}\{\min[\delta_{i-1} d_i (t_i + 1)]\} - 3\rho(\mathbb{C}), \quad i = 1, 2, \cdots, L. \tag{17}$$

Clearly, a design rule for maximizing γ is $\delta_{i-1} d_i (t_i + 1) = \delta_{j-1} d_j (t_j + 1)$ for all i, j.

Fig. 7 presents the maximum obtainable coding gain for the three examples of Section II for various values of R_{eff}. For Example 1 the maximum coding gain is 15.6 dB at the point $R_{\text{eff}} = 3.49$ b/symbol with $K_1 = 14$, $K_2^{(1)} = 206$, $K_2^{(2)} = 78$, and $K_3 = 232$. For Example 2 the maximum asymptotic coding gain is 19.1 dB at the point $R_{\text{eff}} = 3.35$ b/symbol with $K_1 = 386$, $K_2 = 898$, and $K_3 = 66$. A surprisingly large asymptotic coding gain of 20.8 dB is obtained for Example 3 when $R_{\text{eff}} = 3.07$ b/symbol, $K_1 = K_2^{(1)} = K_2^{(2)} = 798$, and $K_3 = 144$.

B. Real Coding Gain of Single-Level Concatenated Schemes

A study of single-level concatenated schemes reveals the possibility of obtaining a remarkably large real coding gain with a very low complexity structure. From a practical point of view, the real coding gain is more interesting than the asymptotic coding gain, especially for an AWGN channel with BER in the range of 10^{-6} to 10^{-9}. The asymptotic coding gain is calculated by ignoring the effect of the error coefficient, by which the reduction of the real coding gain is explained. Forney's rule of thumb [4] enables the computation of the effective coding gain which approximates the real coding gain under certain circumstances at BER in the vicinity of 10^{-6}. Because of the structure of the concatenated coded modulation scheme, this rule of thumb does not hold here. The real coding gain for a desired BER is the difference between the signal-to-noise ratios of the concatenated single or multilevel coding scheme and a reference system at the same value of BER. As a reference system we may use the $2^{R_{\text{eff}}}$– QAM uncoded constellation, for which the probability of error is well known (see, e.g., [17] pp. 278–284).

Consider a structure which consists of a concatenated code at the single partition level and four uncoded bits. Its asymptotic coding gain is bounded, according to (17), by $6 - 3\rho(\mathbb{C})$ [dB]. When the concatenated code satisfies the inequality $d_1(t_1 + 1) \geq 4$, the bound is attained, i.e., the asymptotic coding gain is given (in [dB]) by

$$\gamma = 6 - 3\rho(\mathbb{C}). \tag{18}$$

Example 4: The partitioning is performed by the output bits of a concatenation between an inner binary $(n, n-1, 2)$ single parity check and an outer RS$(N_1 = 2^{n-1}, K_1, D_1)$ code over GF(2^{n-1}). The remaining four bits are uncoded.

Since the error coefficient of the inner code in Example 4 is low, the probability of symbol error at the input to the RS decoder P_{s1} is tightly bounded by the dominant terms of the union bound. The number of codewords at squared Euclidean distance of $d(\mathbb{C}) = 2$ from each codeword does not exceed $A_2 = \left[\binom{n}{2}\right] 2^2 = 2n(n-1)$. Thus, P_{s1} is tightly bounded by

$$P_{s1} \leq A_2 Q\left(\sqrt{d_1 R_{\text{eff}} E_b / 2N_0}\right). \tag{19}$$

The BER at the output of the outer decoder P_{b1} is calculated according to (7) and (19), whereas the BER of the uncoded bits P_{b2} is calculated as the BER of a 16 QAM constellation with minimum squared Euclidean distance of 4. For achieving the highest real coding gain at the BER of 10^{-6}, we use the following design rule: the dimension of the RS code is chosen to obtain $P_{b1} \approx 10^{-6}$ at the signal-to-noise ratio which provides $P_{b2} = 10^{-6}$. This design rule is applied only up to the point where a further increase of the correction capability of the outer code (aimed at lowering the BER) costs, due to the increase of the redundancy, more than the increase of the coding gain. The maximum obtainable real coding gain for Example 4 at BER= 10^{-6} is given in Table I for various values of n. The dimension of the outer code which attains the maximum obtainable real coding gain, for each n, has been chosen according to the aforementioned design rule. Each of the codes listed in Table I has a real coding gain which is very close to the asymptotic [calculated according to (18)] coding gain, due to the small error coefficient of the trivial inner code.

From Table I we deduce that by increasing the length of the code, the normalized redundancy is reduced and, consequently, the real coding gain increases. However, lengthening the code beyond a certain value contributes a negligible increase of the coding gain. This is explained partly by the increase of the error coefficient with an increase of the length. Increasing the length of the code also increases the decoding delay

TABLE I
THE MAXIMUM OBTAINABLE REAL CODING GAIN FOR EXAMPLE 4 AT BER = 10^{-6}

n	A_2	N_1	K_1	R_{eff}	Coding gain in [dB]
5	40	16	10	5	2.95
6	60	32	24	5.25	3.7
7	84	64	50	5.339	3.97
8	112	128	106	5.449	4.3
9	144	256	218	5.5	4.45
10	180	512	440	5.547	4.6
11	220	1024	882	5.566	4.65

and the decoding complexity of the code. For the case of Example 4, increasing the length of the code beyond $n = 10$ yields a negligible increase of the real coding gain.

Replacing the inner code by a code with minimum Hamming distance larger than 2 cannot improve the real coding gain. This follows by observing that such change would increase the normalized overall redundancy, the asymptotic coding gain would reduce according to (18), and the real coding gain is bounded by the asymptotic coding gain. For example, by substituting the binary Hamming $(15, 11, 3)$ code for the binary $(n, n-1, 2)$ code of Example 4, the maximum obtainable asymptotic coding gain is lowered to less than 4.4 dB. Further increase of the Hamming distance of the inner code will cause an additional reduction of the maximum obtainable coding gain.

One may consider the possibility of expanding the single-level concatenated code of Example 4 into a simple two-level concatenated code, by encoding another two or four bits of each channel symbol with RS code. However, the improvement in the maximum obtainable real coding gain turns out to be very small. As an example, let us encode another two bits of each channel symbol with the RS $(2^n, 2^n - 2, 3)$ code over GR(2^n). The increase of the normalized redundancy in this structure (due to the additional code) is negligible, whereas the reduction of the signal-to-noise ratio for which a BER of 10^{-6} is attainable, is significant. By increasing the correction capability of the RS code at the first partition level, the real coding gain at BER = 10^{-6} might increase. However, this improvement is limited by the cost of increasing the redundancy of the RS code at the first partition level. Calculation of the maximum obtainable real coding gain at BER= 10^{-6} for various values of n, indicates a negligible increase over the results of Table I. It is obvious that by increasing the correction capability of the additional RS code, or, alternatively, by encoding the four uncoded bits per symbol of Example 4 instead of two bits, the real coding gain will be reduced.

For achieving larger real coding gain we have to consider more than a single partition level and apply inner codes with larger Euclidean distance. However, such increase of the number of partition levels is accompanied with a steep increase of decoding complexity, unless proper care is taken in the design process. Furthermore, for the case of a rather noisy channel the single-level scheme may prove to be advantageous over multilevel scheme in the sense of nearly optimal coding gain achieved with low constellation expansion and low decoding complexity.

C. Real Coding Gain of Multilevel Concatenated Schemes

This section is concerned with the three examples of Section II. Since the error coefficients of these structures are very large and the BER is calculated at low signal-to-noise ratio, the customary upper bound on the probability of decoding error (obtained by using the dominant term of the union bound) is not tight enough. A computer simulation is performed for calculating the probability of symbol error at the input of the outer decoders P_{si} while the average BER at the output of the outer decoder is calculated according to (7), (8).

The real coding gain strongly depends on the error rates at the output of the inner decoder. Examination of the behavior of these error rates for low signal-to-noise ratios provides some insight into the structure and the design of codes with high real coding gain. The decoding structure shown in Fig. 2 ensures a reliable input to the inner soft-decision decoders from the lower partition levels, due to the outer codes. This reduces the effect of the propagation of errors form one soft decision decoder to the next one (the multistage decoder is suboptimal due, partly, to these effects) and enables us to examine separately the performance of each of the inner decoders.

The inner codes of Examples 1–3 construct lattices according to Construction B [13, ch. 5] with minimum squared Euclidean distance of 8. For maximum likelihood decoding, the error coefficient (named "kissing number" in [13]) is given by [13]

$$\tau = 2n(n-1) + 128A_8(C_1) \tag{20}$$

where $A_8(C_1)$ is the number of codewords of C_1 with a Hamming weight 8. However, due to multistage decoding, the effective error coefficient is larger [12] than the nominal error coefficient. Since the effect of the single check bit code is ignored in the decoding at the first level the effective error coefficient is given by

$$\tau_{\text{eff}} = 2n(n-1) + 256A_8(C_1). \tag{21}$$

The two terms of (21) are the contributions of the binary codes at the first two levels of partition. The remaining bits do not have any effect on the error coefficient, since a change in them may cause in the Euclidean space a change with minimum squared distance of 16. Denoting the contribution of the first level by τ_a and the second level by τ_b (the error coefficient of the first and second partition level, respectively), we have

$$\tau_a = 256A_8(C_1), \quad \tau_b = 2n(n-1), \quad \tau_{\text{eff}} = \tau_a + \tau_b. \tag{22}$$

The values of $\tau_a$and τ_b for Examples 1–3 are listed in Table II.

Fig. 8 exhibits the results of a computer simulation for the output error rates of the inner decoders in most of the relevant range of E_b/N_0, in which a desired average output BER of 10^{-6} to 10^{-9} is obtained for the case of $R_{\text{eff}} = 4$ b/symbol. Pr_{a_i} and Pr_{b_i} for each $i = 1, 2, 3$ stand for the probability of a codeword error at the output of the first, respectively, second level inner decoder in Example i. Examination of Table II and

TABLE II
THE EFFECTIVE ERROR COEFFICIENTS OF THE FIRST AND SECOND PARTITION LEVELS FOR EXAMPLES 1–3

	τ_a	τ_b	τ_{eff}
Example 1	194,304	1,104	195,408
Example 2	53,760	840	54,600
Example 3	7,680	480	8,160

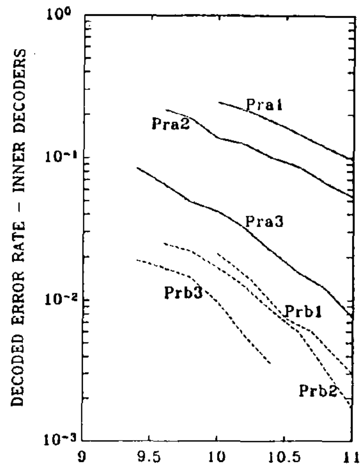


Fig. 8. Results of a computer simulation for outputs error rates of the inner decoders of Examples 1–3.

Fig. 8 clarifies that Pr_a and Pr_b, are increasing with τ_a and τ_b, respectively. This increase is not linear for the relevant range of E_b/N_0, thereby the use of a constant plus an exponential approximation for the error rates is prevented. However, for a proper design of the concatenated multilevel code we have to consider the error coefficients. Thus, the outer decoder at the first partition level for Examples 1–3 ought to be with the maximum correction capability.

The probabilities $P_{s1}(P_{s2})$ are determined by $Pr_a(Pr_b)$, the number of information bits in the binary codeword $k_1(k_2)$, and the length of RS symbol $B_1(B_2)$ (e.g., in Example 3: $B_1 = 2k_1$ yields $P_{s1} = 2Pr_a(1 - Pr_a) + Pr_a^2 \approx 2Pr_a$, and in Example 1: $2B_1 = k_1$ yields $P_{s1} \leq Pr_a$; the exact value is given by the computer simulation). The probability P_{s3} is regarded as the output error rate of uncoded 4 QAM with minimum squared Euclidean distance of 16, which is much smaller than P_{s1} and P_{s2}. Therefore, for concatenated codes with high real coding gain in the BER range of $10^{-6} - 10^{-9}$, there is usually no need for an outer code at the last partition level. A correction capability of one symbol is needed in some cases.

For the design of concatenated multilevel codes with high real coding gain, the correction capability of the outer codes is to be adjusted in order to obtain the same value P_b of average desired BER at all the outputs of the outer decoders. If the BER is not the same at all the outputs of the outer codes, for a desired average BER, then some of the codes contain more redundancy than really needed. Some examples now follow.

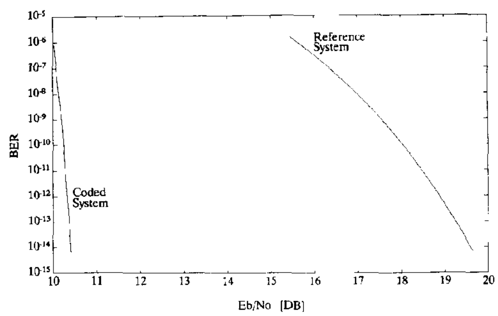


Fig. 9. BER versus E_b/N_0 for Case 6.

Case 1: For the structure of Example 3 with $K_1 = 638$, $K_2^{(1)} = 948$, $K_2^{(2)} = 936$, and $K_3 = 254$ ($R_{eff} = 4.1$ b/symbol), the real coding gain obtained for BER $= 10^{-9}$ is 7.4 dB.

Case 2: The structure of Example 3 is considered, with the outer code at the third level omitted. The dimensions of the outer codes are $K_1 = 752$, $K_2^{(1)} = 960$, and $K_2^{(2)} = 970$, $R_{eff} = 4.23$ b/symbol. The real coding gain obtained for BER $= 10^{-6}$ is 5.6 dB.

Case 3: For the structure of Example 2 with the outer code at the third level omitted and outer codes with the dimensions $K_1 = 376$ and $K_2 = 996$ ($R_{eff} = 4.48$ b/symbol), the real coding gain obtained for BER $= 10^{-9}$ is 7.34 dB.

Case 4: For a structure similar to that of Case 3 but with outer codes (512, 348) and (1024, 992) (R_{eff} = 4.43 b/symbol), the real coding gain obtained for BER = 10^{-6} is 5.6 dB.

Case 5: The maximum real coding gain obtained for Example 1 with BER of 10^{-6} and 10^{-9} is 5 dB and 6.3 dB, respectively. These results are less impressive than those obtainable for Examples 2 and 3. Also, the computational complexity of decoding for Example 1 is much higher. This seems to be mainly due to the large error coefficient of the Leech half lattice (see Table II).

Case 6: Fig. 9 presents BER versus E_b/N_0 for the structure of Example 3 with $K_1 = 754$, $K_2^{(1)} = 966$, $K_2^{(2)} = 958$, and $K_3 = 254$ ($R_{eff} = 4.207$ b/symbol), in comparison with the reference system. The real coding gain for BER of 10^{-6} and 10^{-9} is 5.5 dB and 7.3 dB, respectively. By (17), the asymptotic coding gain is 9.67 dB. Fig. 9 shows that the real coding gain is increasing rapidly with the decrease of the BER (the slope of the reference curve is considerably more moderate) and for small values of BER the real coding gain approaches the asymptotic coding gain. Note that in this case small (relatively to the codes considered in Section III-A) asymptotic coding gain does not imply a small real coding gain.

IV. FURTHER DISCUSSION

We presented and analyzed various concatenated multilevel coded modulation schemes. The codes used are binary inner block codes, concatenated at each level with one or more RS codes. Although the signal constellation considered throughout is 64 QAM, various other constellations could be employed in the same fashion. The simplest scheme considered consists of a single concatenated code, whose output performs a partitioning of the signal constellation, and the remaining bits are uncoded. By using a single check-bit inner code, a real coding gain of 4.65 dB was obtained at BER of 10^{-6}, despite the remarkably low complexity of decoding. Furthermore, it turns out that by using a more powerful inner code the overall performance deteriorates.

An alternative way to describe the structure considered in Example 4 is the following: the inner code is essentially the checkerboard lattice D_n [4], the partition may be viewed as $D_n/2Z^n$, and consequently the RS code is over the field of order $|D_n/2Z^n| = 2^{n-1}$. Extending this idea, one may devise a two-level system based on an n-dimensional lattice partition Λ/Λ', in conjunction with a first-level RS code over a field of order $|\Lambda/\Lambda'|$. For example, let $\Lambda = H_{16}$ [4] and $\Lambda' = 2D_{16}$ where H_{16} is a lattice formed with the aid of the extended $(16, 11, 4)$ Hamming code. Then a RS code over the field of order $|H_{16}/2D_{16}| = 2^{12}$ is applicable. This structure has a twofold advantage, both H_{16} and $2D_{16}$ are simple to decode and the constellation expansion is small. The two level system based on general lattice partitioning deserves further investigation.

Let N_R and N_G be the normalized complexities of decoding, defined as the number of required operations per channel symbol over the real numbers and over GF(2^{10}), respectively. Hard-decision decoding of RS code with correction capability of t symbols requires [18] less than $2t^2$ Galois field multiplications per codeword. As previously shown, the structure of Example 3, with constellation expansion [4] of $2^{\rho(\mathbb{C})} = 3.46$, achieves a remarkable coding gain. Nonetheless, for Case 6 we have $N_R < 8$ (decoding of Λ_{16} according to [12]) and $N_G = 7.9$. With the Leech lattice an asymptotic coding gain of 6 dB but (due to very high error coefficient) considerably smaller effective coding gain are obtained. The lowest average normalized complexity of maximum likelihood decoding of the Leech lattice is about $N_R = 417$ [15], whereas $N_R = 167$ for the suboptimal decoding of Forney [12]. Optimal decoding of the trellis codes of [4], with asymptotic coding gain of 6.02 dB and effective coding gains of 4.76 and 5.57 dB, requires $N_R = 584$ and $N_R = 3592$, respectively. Example 3 demonstrates a better performance, despite the smaller number of operations required for decoding.

The construction of a concatenated multilevel coded modulation scheme was performed in the following fashion. We employed familiar lattice codes as inner codes. Following the selection of the inner codes and the number of bits per RS symbol, the length of the outer codes is selected, for simplicity, to be a power of 2. A computer simulation for the output BER of the inner decoder was then performed for determining the number of information symbols per outer codewords. These numbers were selected such that the output BER of each outer decoder is equal to the same desired value. The overall design was aimed at achieving large real coding gain and low complexity of decoding.

The decoder of Fig. 2 suffers from a relatively long delay. The decoding delay is determined mainly by the length of the RS codewords, the number of partition levels and the

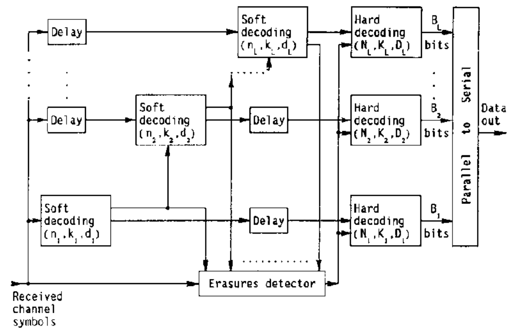


Fig. 10. Decoding scheme with side information.

implementation of the RS decoders. For Example 3 the delay exceeds $2 \cdot 1024 \cdot 10 + 256 \cdot 8 = 22528$ time units. A decoding structure with reduced decoding delay but larger memory size is depicted in Fig. 10. It retains a good performance, provided the component codes are appropriately selected. The decoder of Fig. 10 consists of an inner decoder (a conventional lattice decoder), an erasure detector (for obtaining side information) and L outer decoders. Since all the outer decoders are set in parallel, the overall delay is determined by the longest outer code, rather than the sum of the delays of L outer decoders as in Fig. 2. However, the memory size in Fig. 10 is the sum of the memories of the L outer decoders, whereas the memory size required by the structure of Fig. 2 is about the memory space needed for the largest RS codeword.

It is beneficial to contrast the performance of our scheme with those obtained by other concatenated coded modulation schemes. A concatenation between an inner 16-state trellis code over 8-PSK constellation and an outer RS code is presented in [9]. At $R_{\text{eff}} = 1.874$ b/symbol, coding gain of 6.5 dB at BER $= 10^{-9}$ and 4.9 dB at BER $= 10^{-6}$ are achievable. A concatenation between an RS outer code and simple binary block codes combined with 8 PSK constellation, is presented in [10]. By a particular structure of this kind, employing a RS $(255, 223)$ code with overall effective rate of 1.75 b/symbol, coding gains of 4.94 dB at BER= 10^{-6} and 7.02 dB at BER $= 10^{-10}$ were achieved.

ACKNOWLEDGMENT

The authors are grateful to the referees for their careful reading of the manuscript and for their useful suggestions that helped to improve the paper.

REFERENCES

[1] G. Ungerboeck, "Channel coding with multilevel/phase signals," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 55–67, Jan. 1982.
[2] A. R. Calderbank and J. E. Mazo, "A new description of trellis codes," *IEEE Trans. Inform. Theory*, vol. IT-30, pp. 784–791, Nov. 1984.
[3] G. D. Forney, Jr., *et al.*, "Efficient modulation for band-limited channels," *IEEE Select. Areas Commun.*, vol. SAC-2, pp. 632–647, Sept. 1984.
[4] G. D. Forney, Jr., "Coset codes—Part I: Introduction and geometrical classification," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1123–1151, Sept. 1988, part II.
[5] ——, "Coset codes—Part II: Binary lattices and related codes," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1152–1187, Sept. 1988, part II.

[6] V. V. Ginzburg, "Multidimensional signals for a continuous channel," *Problemy peredachi informatsii,* vol. 20, no. 1, 1984 (translated from Russian by Plenum).

[7] A. R. Calderbank, "Multilevel codes and multistage decoding," *IEEE Trans. Commun.,* vol. 37, pp. 222–229, Mar. 1989.

[8] G. J. Pottie and D. P. Taylor, "Multilevel codes based on partitioning," *IEEE Trans. Inform. Theory,* vol. IT-35, pp. 87–98, Jan. 1989.

[9] R. H. Deng and D. J. Costello, Jr., "High rate concatenation coding systems using bandwidth efficient trellis inner codes," *IEEE Trans. Commun.,* vol. 37, pp. 420–427, May 1989.

[10] T. Kasami, T. Fujiwara, T. Takata, and S. Lin, "A concatenated coded modulation scheme for error control," *IEEE Trans. Commun.,* vol. 38, pp. 752–763, June 1990.

[11] H. Imai and S. Hirakawa, "Multilevel coding method using error-correcting codes," *IEEE Trans. Inform Theory,* vol. IT-23, pp. 371–377, May 1977.

[12] G. D. Forney, Jr., "A bounded-distance decoding algorithm for the Leech lattice, with generalizations," *IEEE Trans. Inform. Theory,* vol. IT-35, pp. 906–909, July 1989.

[13] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups.* New York: Springer-Verlag, 1988.

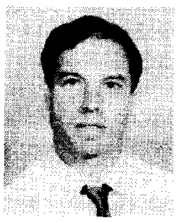
[14] F. J. Macwilliams and N. J. A. Sloane, *The Theory of Error Correcting Codes.* New York: North Holland, 1978.

[15] Y. Be'ery, B. Shahar, and J. Snyders, "Fast decoding of the Leech lattice," *IEEE J. Select. Areas Commun.,* vol. 7, pp. 959–967, Aug. 1989.

[16] J. M. Wozencraft and I. M. Jacobs, Principles of Communication Engineering. New York: Wiley, 1965.

[17] J. G. Proakis, *Digital Communications.* New York: McGraw-Hill, 1989.

[18] R. E. Blahut, *Theory and Practice of Error Control Codes.* New York: Addison-Wesley, 1983.



Hanan Herzberg was born in Israel, in 1958. He received the B.Sc and M.Sc degrees in electrical engineering with distinctions from the Tel-Aviv University, Israel, in 1984 and 1987, respectively. He is presently working toward the Ph.D degree in electrical engineering.

From 1984 to 1988 he was with the Research and Development Department, Communication Division, Tadiran Ltd., Israel, where he is currently a consultant for a digital communication and digital signal processing projects. Since 1988 he is also a teaching assistant at the Tel-Aviv University. His current research interests include coding theory, digital communication, and digital signal processing.

Yair Be'ery (M'87) was born in Tel-Aviv, Israel on July 6, 1956. He received the B.Sc., M.Sc., and Ph.D degrees, all in electric engineering, from Tel-Aviv University, Tel-Aviv, Israel, in 1979, 1979, and 1985, respectively.

In 1979–1985, he was a Senior Research Engineer in the Research Laboratories, Israel Ministry of Defense, and was involved in research and development of digital signal processing systems and design of VLSI architectures for parallel processors. Since 1985 he has been with the Department of Electrical Engineering—Systems, Tel-Aviv University, Israel, where he is a Senior Lecturer. In 1990–1991 he was with The DSP Group, Givat-Shmuel, Israel, as a Director of DSP Architectures, on sabbatical leave from Tel-Aviv University. He has also been a consultant to several companies in Israel, which include Tadiran—Telecommunications Group, RAD Data Communications, ECI Telecom, Elisra Electronic Systems, and The DSP Group. He received the Eliyahu Golomb Award from the Ministry of Defense, Israel, in 1984, and the Rothschild Fellowship for postdoctoral studies at the Rensselaer Polytechnic Institute, Troy, NY, in 1986. His research interests include error control coding, combined coding and modulation, VLSI architecture for systolic arrays and neural networks, and adaptive algorithms for speech and data transmission.

Jakov Snyders received the B.Sc., M.Sc., and D.Sc. degrees in electrical engineering from the Technion—Israel Institute of Technology, Haifa, in 1966, 1969, and 1972, respectively.

Since 1974 he has been with Tel-Aviv University, Tel-Aviv, Israel. He has held visiting positions at the University of Toronto, University of California-Los Angeles, McGill University, Montreal, P.Q., Canada, and the Rensselaer Polytechnic Institute, Troy, NY. His research interests include error control coding, spread spectrum and multiple user communication, and stochastic processes and systems.