Information-Theoretic Approach to the Analysis of Trellis Complexity of Lattice and Nonlattice Periodic Packings

Ilan Reuven and Yair Be'ery

Department of Electrical Engineering - Systems
Tel-Aviv University, Ramat Aviv 69978, Tel-Aviv, Israel
E-mail: ybeery@eng.tau.ac.il

Abstract — **The trellis representation of lattice and nonlattice periodic packings is studied from a new perspective. We concentrate on the so-called fundamental module of the packing which extracts the dynamical component of the packing. We use this framework to define the trellis construction of nonlattice periodic packings. From this viewpoint we extend the notion of entropy/length profile (ELP) to periodic packings by introducing the entropy/dimension profile (EDP). We use these profiles to derive lower bounds on the trellis complexity of periodic packings.**

I. Notations and Definitions

An n-dimensional lattice is called *orthogonal lattice* if it is a direct sum of n one-dimensional lattices $\{l_i\mathbb{Z},\ 1 \le i \le n\}$. We denote this lattice by $\boldsymbol{l}\mathbb{Z}$, $\boldsymbol{l}\mathbb{Z} \triangleq l_1\mathbb{Z} \oplus l_2\mathbb{Z} \oplus \ldots \oplus l_n\mathbb{Z}$.

Definition 1: A *periodic packing* P is a union of a finite number of translates of an n-dimensional orthogonal lattice $\boldsymbol{l}\mathbb{Z}$. Let Q be a set of *glue vectors* (translate representatives), then we have $P \triangleq Q + \boldsymbol{l}\mathbb{Z}$, that is, $P = \{q + \boldsymbol{l}\mathbb{Z} : q \in Q\}$.

All useful constructions of nonlattice packings comply with the above definition of periodic packings.

Definition 2: Let P be a periodic packing and let $\boldsymbol{l}\mathbb{Z}$, $\boldsymbol{l} \triangleq \{l_i,\ 1 \le i \le n\}$ be its (densest) orthogonal sublattice in the given coordinate system. The points of the packing which are included in the polytope $T = \{x: \boldsymbol{x} = \boldsymbol{a} + \sum_{i\le n} \theta_i \cdot l_i \cdot \boldsymbol{e}_i,\ 0 \le \theta_i < 1\}$, will be called a *fundamental module* of the periodic packing, where $\boldsymbol{a}$ is an arbitrary point in $\boldsymbol{R}^n$, and $\boldsymbol{e}_i$ is a unit vector along the ith coordinate. We will henceforth denote by the pair $(Q,\boldsymbol{l})$ the fundamental module and the distance set of the underlying orthogonal sublattice.

In a given coordinate system, i.e., for a given fundamental module of a periodic packing P, a trellis diagram for P is defined by the ordering of these coordinates into a nested sequence of vector spaces. We order the n one-dimensional spaces and we denote them by $\{W_i,\ 1 \le i \le n\}$. We define the nested sequence of increasing subspaces

$V_0 = \{0\}$, $V_i = V_{i-1} \oplus W_i$, $1 \le i \le n$. Clearly, $V_0 \subset V_1 \subset \ldots \subset V_n = \boldsymbol{R}^n$. We also denote $U_J \triangleq W_{j1} \oplus W_{j2} \oplus \ldots \oplus W_{j|J|}$, where $J = \{j_1, j_2, \ldots, j_{|J|}\}$. The projection $P_J(x)$ of a vector $\boldsymbol{x} \in \boldsymbol{R}^n$ onto the $|J|$th-dimensional space U_J is defined by $P_J(x) = (x_{j1}, x_{j2}, \ldots, x_{j|J|})$. We denote by I the index set, $I \triangleq \{1,2,\ldots,n\}$ and $i^- \triangleq [1,2,\ldots,i]$, $i^+ \triangleq [i+1, i+2, \ldots, n]$.

In the sequel we also use the information-theoretic measures $H(X)$ - the entropy of the ensemble X, and $I(X;Y)$ - the mutual information between a pair of ensembles X and Y. All the logarithms here onwards are assumed to be taken to same arbitrary base.

II. Entropy / Dimension Profiles

Let P be a periodic packing which incorporates a fundamental module $(Q,\boldsymbol{l})$. We make Q into an ensemble whose sample space is the vector points Q. Let M denote the cardinality of Q. We assign each point of the sample space a uniform probability of $1/M$ and we denote by X_J a random $|J|$-tuple variable that takes on the values of the set $P_J(Q)$ with probabilities that are determined by a uniform distribution of the elements of the fundamental module.

In what follows we assign each coordinate an arbitrary weight w_i, $1 \le i \le n$, and we evaluate some profiles with respect to the chosen weight set $\boldsymbol{w} = \{w_i,\ 1 \le i \le n\}$. All profiles are of a packing P with a fundamental module $(Q,\boldsymbol{l})$.

Definition 3: The *entropy/dimension profile* (*EDP*) will be defined as the sequence $\boldsymbol{h}'(P,\boldsymbol{w}) = \{h'_i(P,\boldsymbol{w}),\ 0 \le i \le n\}$, where $h'_i(P,\boldsymbol{w}) \triangleq \min_J \{H(X_J) + \Sigma_{j\in J} w_j : |J| = i\}$.

Definition 4: The *conditional entropy/dimension profile* (*conditional EDP*) is the sequence $\boldsymbol{h}(P,\boldsymbol{w}) = \{h_i(P,\boldsymbol{w}),\ 0 \le i \le n\}$, where $h_i(P) \triangleq \max_J \{H(X_J|X_{I\text{-}J}) + \Sigma_{j\in J} w_j : |J| = i\}$.

III. Lower Bounds on State Complexity

Theorem 1 (*state complexity profile in an ordered coordinate system*): $s_i(P) \ge I(X_{i^-}; X_{i^+})$, $0 \le i \le n$. Lattice packings meet this bound with equality. □

The proof of this theorem is based on Theorem 4 of [2].

Theorem 2 (*state complexity profile in a given (unordered) coordinate system*): $s_i(P) \ge h'_i(P,\boldsymbol{w}) - h_i(P,\boldsymbol{w})$. □

Any weight set yields a valid bound. Particularly, for the weight set $\boldsymbol{w} = \{-\log \boldsymbol{l}\}$, we get

$$s_i(P) \ge -\log V(P) - h_i(P, -\log \boldsymbol{l}) - h_{n-i}(P, -\log \boldsymbol{l}),\quad 0 \le i \le n,$$

where $V(P)$ is the volume of the packing.

This bound extends the DLP bound of [1] and the ELP bound of [2] to codes whose symbols are not drawn from the same alphabet set.

Corollary 3: The state complexity profile of a periodic packing P in $\boldsymbol{R}^n$ in any coordinate system (wherein it admits a periodic structure) is bounded by

$$s_i(P) \ge (n/2)\cdot\{\log \gamma(P) - (i/n)\cdot\log \Gamma_i - (n\text{-}i)/n\cdot\log \Gamma_{n-i}\},$$

where $\gamma(P)$ is the coding gain of the packing and Γ_j is the maximum coding gain of any j-dimensional packing. □

This bound extends a similar result of [1] to nonlattice periodic packings.

Further bounds, properties of the different profiles, and corresponding bounds on the branch complexity of periodic packings are given in [3] and will be presented in this talk.

References

[1] G. D. Forney, Jr., "Density/length profiles and trellis complexity of lattices," *IEEE Trans. Inform. Theory*, vol. 40, pp. 1753-1772, Nov. 1994.

[2] I. Reuven and Y. Be'ery, "Entropy/length profiles, bounds on the minimal covering of bipartite graphs, and trellis complexity of nonlinear codes," *IEEE Trans. Inform. Theory*, vol. 44, pp. 580-598, Mar. 1998.

[3] I. Reuven and Y. Be'ery, "Entropy/dimension profiles, the weighted coordinates bound, and trellis complexity of periodic packings," submitted to *IEEE Trans. Inform. Theory*, Sept. 1997.